MATHEMATICS TEACHING

THE BULLETIN OF THE
ASSOCIATION FOR
TEACHING AIDS IN
MATHEMATICS

★

No. 12—MARCH 1960 Price 3/6

MATHEMATICS TEACHING

Published three times yearly

Editor:

CLAUDE BIRTWISTLE,

1 Meredith Street, Nelson, Lancs.

Book Review Editor:

CYRIL HOPE

68 Malvern Road, Powick, Worcs.

Contributions to the journal and enquiries for advertising space should be addressed to the Editor. Contributors are asked to keep mathematical notation as typographically simple as they can, and to submit illustrations, where necessary, in a form suitable for reproduction.

Enquiries about distribution should be addressed to the Secretary of the Association.

Coulton & Co. Ltd., Printers, Nelson and Colne.

MATHEMATICS TEACHING

CONTENTS

No. 12 March, 1960 3/6

EDITORIAL

THE SIXTH FORM

Between thirty and forty per cent of candidates sitting an Advanced level subject in the G.C.E. fail the examination; this fact is common knowledge. But other statistics relating to work at this level are less well-known and should receive more attention. For example, according to the H.M.S.O. report *Education in* 1958, of 55,000 pupils who left school aged seventeen and over in that year, about 21,000 did not possess any 'A' level passes in the G.C.E.

This figure merits closer examination for it may be argued that pupils aged seventeen are not necessarily true Sixth Form pupils. To some extent the figures support this view, since of this total, 18,000 did not offer an 'A' level subject at all. Yet surely pupils of this age should have entered a Sixth Form, imbibed something of its atmosphere, have glimpsed something of the fund of knowledge that lies ahead, have begun to understand the historical and philosophical background of their work, and have learned to appreciate the satisfaction of critical study. Yet, assuming that they have had this experience, 21,000 have left school with nothing to show for it in a material sense; they were judged, either by their teachers or the examiners, to be unworthy of a certificate of competence in advanced-level study.

Further reference to the statistics in this booklet reveals other interesting facts about these school leavers. Thus we find that only 14,295 of the 55,000 leavers enter University, a further 8,491 go to Teacher Training Colleges, and the rest—over 32,000—go direct to employment or to other types of further education. (It may be of interest to quote the corresponding figures for leavers who had passed mathematics at 'A' level: of 9,170 leavers, 6,290 went to University, 430 to Training College, and 1,570 direct to employment). In other words, well over half of our seventeen and eighteen year old school leavers are not proceeding to full-time academic study. This is a development of recent years and should be a prime factor in our consideration of Sixth Form studies; yet it is too often overlooked. Do we remember these people when framing our curricula? Do we really cater for *their* needs?

In this issue we publish two reports which touch on this problem. The first, challengingly called by its author *Is Sixth Form teaching good enough?*, summarizes the views from various University Mathematics Departments on the quality of students entering University. The substance of their criticism is clear enough: students leaving school have an effective mastery of technique but an inadequate appreciation of principle. This speaks ill for Sixth Form teaching; are we as teachers accepting the easy way? It is easier to teach techniques than it is to impart the principles of a subject, and yet paradoxically—and this may be the root of the problem—provided they are properly taught, it is easier to *learn* underlying principles than it is to master techniques.

The second report is on an international conference held in August last dealing with *Transition from Sixth Form to University*. We find the theme repeated here, but this time the blame is attached to the examiner: the question papers set in this country tend to seek 'recognition of particular tricks' rather than appreciation of mathematical principle. But the teachers do not escape criticism, for it is considered

that too few concern themselves with the learning process; they may know what they teach, but they do not understand how to teach what they know. However, one of the most significant parts of this report as related to Sixth Forms in this country is where it is stated that our examination system imposes on all our students a particular type of academic syllabus suited—and originally intended—for one type of specialist only

There is a basis in history for our present position. By tradition our Grammar Schools have been the gateway to the Universities and the examinations taken at the end of grammar school courses became, under the influence of matriculation requirements, a test of ability to proceed to University studies. Professional organizations, however, seeking to lay down a standard of academic achievement for entry to their own qualification system, decided to take the matriculation standard as their own, notwithstanding the fact that the standard and type of examination were determined by factors which might differ from their particular requirements. Consequently a test of academic studies such as is the natural introduction to a course of study at a University has become the assessment of having completed successfully a course of secondary education. This brings us back to the point which was made initially above: is the aim and curriculum of our Sixth Form really suited to the present-day requirements? More specifically, are we fitting our education to the child?

Not all secondary school pupils, nor all Sixth Formers, are potential University students, especially in this country where the Universities have always emphasized literary and pure scientific studies at the expense of the appled sciences. There are always those who 'think with their hands', the practical people who become our engineers and our industrial administrators; there are those who seek to create through their hands, the artists and craftsmen; there are those who are interested in people and human relationships and seek to find this in such work as nursing and social welfare work; and there are many more similar people. They share one thing: the need for the practical approach of day-to-day realism in their work and studies. Not for them the abstraction of a pure mathematical problem, or the thought behind a literary masterpiece, or the analysis of a master historian. For these are the people who live in their present and immediate world, and the appeal of pure academic studies is often not merely uninspiring but distasteful to them.

The position is aggravated by the fact that increased numbers of students are now entering the Sixth Forms. This is partly due to the encouragement to stay on at school, to the approaching bulge in the school population, and (a new factor now appearing) to the inability of the labour market to absorb leavers at sixteen years of age. A large number of experienced teachers of Sixth Form students are finding themselves between two opposing pressures: on the one hand is the increasing number of students with their wider abilities and diversified interests; at the same time the Universities are pressing for higher standards from their entrants. These demands of the Universities have their effect, through examinations, on all students whether they intend to go to University or not. Small wonder, therefore, that many Sixth Form teachers are looking questioningly at their work and at the examinations for which they are required to prepare their pupils; it is appropriate that they should, if only because of the fact that the Sixth Form has so long resisted change. But what solution shall they seek?

The first essential is a break with the traditional type of syllabus; the second is to bring the work in school into relation to the intended careers of the students. The next requirement is a reform in teaching method, and finally there is need for an examination which will test the comprehension of the basic ideas of the subject over a general field and also test ability in a more specialized field of study which the student is able to choose according to his interests. A diversity of interests and vocational aspirations should produce a corresponding diversity of treatment. Thus in the mathematical Sixth we find the mathematics specialist who intends to take an honours degree in the subject as well as a number of students who are hoping to follow occupations where the study of mathematics in some specialist form is an essential part of the work, e.g. statistics. Others are looking to vocations where mathematics is an important tool in the work being done, e.g. all types of engineering. Finally we have the student who wishes to study the subject as part of a general education, e.g. the student going to Training College not intending to specialize in mathematics. One syllabus, one method of approach, one examination cannot successfully meet the requirements of all these different interests. Admittedly it is not easy for the teacher to diversify syllabuses and methods of treatment in a Sixth Form class including these many elements in what is often a small teaching unit. But in order to diversify, one must first unify by finding a common denominator of what is essential to all taking a Sixth Form course, and from this starting point construct a syllabus based on modern thought and usage of mathematics. Its aim should be to show the essential philosophy of the subject, to outline, the principles of advanced mathematics, to demonstrate modern mathematical methods and show the application of the subject to the problems of a technological age. Much that is found in current 'A' level syllabuses contributes little to any of these ends. A recent report on the suitability of G.C.E. 'A' level syllabuses as preparation for University studies* suggests that many existing topics be omitted, and among other suggestions calls for the early introduction of the Algebra of Sets as an 'A' level topic. Again, the A.E.B. has seen fit to break away from the traditional syllabuses by introducing papers on special aspects of mathematics suited to the interests of students other than science specialists. These are moves in the right direction.

In his report on the International Conference, Mr. Fletcher makes it clear that the standard of work in this country does not compare with some that is found abroad. Many will seek to place the blame specifically, but *all*—the universities, the examiners, and the sixth form teachers—have a share of that blame and a responsibility for effecting an improvement. The urgent need is for the teacher in the classroom to adapt his teaching to the current requirements of his students, for a reform here could initiate a new outlook on mathematical study in this country. It is our Sixth form teachers who can bring to mathematics teaching the new thought and inspiration which it has lacked in recent years.

*Report of an Enquiry into the Suitability of the G.C.E. Advanced Level Syllabuses in Science as a Preparation for Direct Entry into First Degree Courses in the Faculty of Science: University of Birmingham, 1959.

Presidential Address to the Association

WHAT MATTERS MOST

CALEB GATTEGNO

(Owing to absence abroad our retiring President, Dr. Gattegno, is unable to be present to address the Annual General Meeting at Blackpool. Consequently, he was invited to write an address for this issue of 'Mathematics Teaching'.)

The main tasks confronting teachers of mathematics, and more especially, those in Great Britain, will be the subject of this, the first Presidential Address of the Association. Even if we did nothing to meet the challenges of the present and the future, no-one would blame us, but I shall assume here that quite a number of mathematics teachers, at all levels of instruction, have uneasy consciences and are prepared to examine their work critically.

Those who have decided to give their energy and their time to the meeting of these challenges, already know that they are engaged in a creative work of great importance. Teaching can be transformed, as can any other field of experience, into an area in which the creative qualities of the mind find their place and use. When this happens teachers feel that the reality they are meeting is larger than themselves, is inspiring and rewarding. It is extraordinary to find that so few teachers know that the situation in which they live and operate is endowed with such qualities.

We are in need of as many inspired teachers as there are classes. To these classes children are sent without their consent and are submitted to whatever we are as persons and what we can do with them. Only if we really care for them, and know what we are doing to them, do we justify our relation to the future and to the education of the next generation.

Are we, on the whole, in the profession, to go on believing so many current opinions that are neither founded in fact nor revealed truths?

Are we always to leave to someone else the responsibility of telling us what to do and how to do it?

Are we to close our eyes to all realities that do not at once fall into the patterns to which we are accustomed?

Are we to be content with a change in emphasis, or a new shade here and there, even if we feel that an overall transformation is needed?

Are we to continue to think in terms obviously inadequate to the challenges, or are we to attempt to gather the necessary skills and knowledge needed to illuminate our paths?

At present, as a profession, we are dismally ill-equipped. The Training Colleges and Departments of Education are doing what they can, but that is reduced to little because the necessary knowledge for meeting the challenges is either not yet available or is considered unimportant.

It is in order to answer some of the questions above, and to define some others that need consideration from the best among us, that I choose this theme for my address.

We must stop believing that the bright children are those who are verbally inclined and who manage to get into Public Schools (if they have the means) or into grammar schools. Brightness is a complex quality, and one dimension of it is emotional. If children are given what corresponds to their make-up, they become bright. If they are denied it, they give up the struggle to make sense of so many words. So that *we* have a share (and I think it is an important one) in the formation of the legions of the less bright pupils because of our preconceived ideas of how we should carry out our job of teaching. These need re-examination at once.

A.T.A.M. has tried to concern itself with ways of doing things, with reaching pupils who usually shy away from words, and has questioned the entirely verbal approach to mathematics. In contrast to that attitude, as far as I can tell from my experience, the great majority of teachers are still doing in the classroom what was done centuries ago, unquestioningly following the syllabus and the suggestions of someone else. We must stop thinking that there exists a person hidden somewhere who knows exactly what each one of us has to do in his class at such-and-such a moment. The responsibility for teaching would seem obviously vested in the teacher who is at that moment there. But, in fact, it does not work out in that way. There is the syllabus to cover, the many pages of the textbook to turn, the demands of several shadowy persons and institutions to satisfy *before* we consider the concrete situation of our class. The failures in the shadowy regions we interpret as our pupils' or our own when, clearly, all that is at fault is an attitude: *our loyalty*. This we owe, as teachers only to truth and to the future. Big words!—but without what they cover we are nothing; or worse, we are frauds.

We must learn to take our share, fully, if possible, in the educational set-up. That is very important. We can satisfy inspectors, parents and examiners more surely by being part of reality, than by allowing all sorts of illusions to come in. (One of them can be singled out: the equating of 'covering the syllabus' and 'our pupils have mastered its content').

Only when we know better how to teach can we expect to lead our pupils towards mastery; we shall only know that when we know how our pupils learn. It is there that our main task is to be found. *We have to study learning* as a matter of course and all the time: not only in books, but where it takes place: in front of us, in the classroom. Then we shall drop that other illusion of believing that each question has an answer. It may have; but, more likely, if it is a hard question it will require study and sleepless nights, and will only lead to one of many possible approximate solutions. Life questions are not as carefully schematic as the ones we meet in the textbooks—particularly of mathematics. Teaching questions are complex and *we have to learn to think in a complex way about complex questions.* The other way is illusion again.

In our reality, people are involved. People are moody and unpredictable, demanding that we meet them without preconceived ideas. They want to be interested and challenged, and they dislike being taken for 'things', regimented, and their individuality forgotten. (Of course, some seem to like it. I call them anti-social because they have agreed to conform to what has emptied them of all their will to be themselves). Because we are dealing with people, teaching is a great art and we all should see it as such. It is a difficult art that requires us to work hard at it to be good at it. This is just the opposite of what teachers believe when they begin—and

often even when they retire, for they, too, can be soulless as a result of 'good' behaviour for so many years.

Teachers must resolve to accept that only they, in their classes, at each instant, will find what to do that is right in the situation. No book, no lectures, can tell them, once for all, what to do in all cases. If we are honest, we must admit that our ideas about teaching are not as articulate and full as they should be. We have cliches and slogans, and use them freely.

We are frequently resistant to new ideas and new methods. If someone comes with a proposal, and if it has not got final shape (a syllabus with suggestions, or even textbooks, examination questions, etc.), we do not want to waste time looking at it. In this way we so often lose a very good opportunity of improving ourselves and of giving better service. To close our eyes to reality because it appears in unfamiliar garments is very general. But can teachers afford that, since they do not work ultimately for themselves, but for children and the future? If teachers became sensitive to the uniqueness of their function and of its importance, they would look for what would help them to improve their work, and they would accept responsibility for using it in their own way.

Techniques are nothing if that background of sensitivity is not there; but they are everything with it. For then one can reach that level of awareness where one is able at once to translate into action what is known to be for the good of the students —for them as individuals and for their future life.

The techniques I think of are those I know personally; my readers will think of others. But we must all be agreed that what matters most is that we reach a level of confidence based on knowledge, so that we can stand up for what we think and do, not because we have faith, but because it is true.

That is possible if we deliberately and with all the necessary energy pursue the knowledge needed in psychology, in mathematics, in methodology, and in experimental teaching. Not all will pursue everything; not all will need to. As an Association we can pool our experience and share it. That was the reason for its being formed.

In psychology, we are primarily interested in knowing what is the process by which mathematical experience is acquired; what makes it mathematical and nothing else; what are the various ways by which the mind creates those dynamic mental structures that become a power when one possesses them. We also want to know what is creating obstacles, which factors can be dispensed with, and which are indispensable (e.g. is sight necessary?). We want to know, too, if we can substitute one experience for another, and whether special techniques are needed to make the substitution efficient. (How far are words part of the mathematising process? Is there a mathematical thought that exists outside all the uses of senses, language, signs? Can we attain it? Do we need to know of it to reach the handicapped children properly? What problems do we meet in these people whose universe of experience differs from ours?) We want to know how long is needed for such-and-such a mind to reach mastery of this or that field of mathematical experience, and whether personality, language, and the current environmental modes of thought have parts to play.

These are some of *our* psychological problems, and we can all help each other by finding parts of the answers and by commissioning the specialists whose additional help is required.

In mathematics, our needs as teachers are not identical to those of scientists and technicians. What we need most is an understanding of the mathematical thinking processes (which are not reducible to deductive reasoning, as logic is only a section of mathematics and is only used in some stages), and the overall picture of the field with examples of sufficiently varied mathematical behaviours to be acquainted with the historic movements and their contributions. We need to learn to transfer to our pupils the power that mathematics gives over reality, not only the practice of some skills.

The curriculum for teachers in Universities and Colleges must be altered to fit present needs. Our duty to the future demands that teachers have an insight that differs from that of scientists and technologists. We educate through mathematics, and that means opening new vistas, taking the student to higher levels which can no longer be equated with Algebra, Geometry, Trigonometry and Calculus (all at least 300-year-old subjects). We have to understand the way that mathematicians become aware of structures; which are the important ones; how they can be made to act upon each other to provide classes of relationships about which this or that can be said. The truth of any mathematical statement depends on the fact that everyone can ensure for himself that the statement is contained in (is deducible from) the set of relationships used to introduce the situation. As teachers, we need to know how to classify the mathematical species so that they can be recognised and the appropriate statements made immediately. We need to know when a situation requires that we use topological or algebraic arguments; how to disentangle the components of a complex entity and take steps to improve our grasp of it.

This sort of curriculum is being developed in some places, and it is possible to become acquainted with it by reading, for example, Professor G. Choquet's course of analysis or the latest publications in the U.S.A. We in England have little to show. What is needed is the continuous development of a sensible approach valid throughout the whole course of school education, replacing the present bottleneck resulting from all sorts of influences dating from various times.

We need to know how to present our work to those learning mathematics. In fact, we need a *methodology of the methodology of mathematics teaching*.

The teaching of mathematics has one aspect that relates to the classroom, that being its methodology; but each science has developed, besides its results, an interest in how these results are obtained, what the problems of the science are, how they differ from those in others, etc. This part is the one that has, so far, been so utterly neglected in the methodology of teaching.

When we know why we do something in the classroom and what effect it has on our pupils, we shall be able to claim that we are contributing to the clarification of our activity as if it were a science. One aspect of this, at least, has been developed a little; I mean the study of children's mistakes and what we can learn from them. This is one of the most fascinating fields of creative work for teachers, since they are all the time confronting the teaching reality and are the people most interested in seeing clearly in it.

Another aspect of our work could be the study of the mathematical possibilities of children. So much here is sheer prejudice at present. Indeed, it is now becoming known that children's mathematical abilities are incomparably greater than has

previously been suggested. Much time, energy and frustration could be avoided if teaching were related to children's thinking powers.

My last point will be concerned with *experimental teaching*. In our Association this theme has often been touched upon. We do not want anybody to share our views until these have been proved right in the classroom. We do not think that we are entitled to anyone's hearing unless we have assured ourselves in advance that we ask for his attention on the grounds of our successful experience. If all teachers demanded this standard of behaviour from all their advisers, it could contribute greatly to making the profession a responsible one, where words mean exactly what they say. Experimental teaching is the use of the classroom in order to learn something worth communicating about the activity of teaching, so that improvements can follow in the work of others who are facing similar circumstances.

In conclusion, I want to say that *what matters most* is that we stop being the toy of prejudice and cliche, and seriously embark upon doing our work as if it were the most precious activity we have. By becoming aware of the various components of our professional life, we shall at the same time give ourselves exciting and exhilarating moments, make a contribution to science, help the young generation to meet its future, and, more than anything, live on the side of truth.

New Delhi,
October, 1959.

TEACHING AIDS AND LOGIC

II - LOGICAL MODELS

W. SERVAIS

(Translated by C. A. Winyard)

Introduction

(W. Servais is President of the Belgian Society of Teachers of Mathematics and Secretary of the International Commission for the Study and Improvement of the Teaching of Mathematics. This article, in a translation by Mr. C. A. Winyard, is reprinted from *Documentation No.* 5, a publication of the Belgian Ministry of Public Instruction, with their kind permission. The previous article in this series showed how common tricks and puzzles can lead to an appreciation of logical structure. This article provides an introduction to the formal symbolism of the modern algebra of logic for those who are unfamiliar with it, and also gives details of a number of demonstration models which can easily be made from simple electrical components.

(We hope that later articles will show further applications of the algebra and give more advanced models to build).

Mathematics is a school of logic in which one learns by practice to:

state a concept precisely and define it;
form and express a judgement;
follow a train of reasoning and test its validity;
invent, put into shape, and criticize a reasoned argument;
set a problem, solve it, and discuss it exhaustively, considering all possible cases.

With geometry as an example mathematics makes plain the structure of a deductive theory in which theorems are built on the foundation of accepted axioms and technical terms are explained by definitions.

The logic required in mathematics is not restricted to that of Aristotle, which reaches its highest point in the study of syllogisms. Even elementary mathematics makes use of the logic of relations, the most important modern contribution to logic.

In their day Descartes and Pascal referred to geometrical proofs as models of reasoning. A century ago, two mathematicians, Boole and De Morgan,[1] made real Leibniz's dream of 'discovering a universal algebra in which the laws of deduction should be the rules of calculation'. Symbolic logic has developed far more in a hundred years than Aristotelian logic has in two thousand. In its modern form, logic is one of the primary branches of mathematics. The contributions made by Frege, Russell, Whitehead, Hilbert, and so many others to the study of the foundations have opened up a more abstract way of conceiving the axiomatic basis of mathematics.

If doing mathematics is to give training in logical reasoning, there must be full awareness of the rules of logic on which the arguments are based. Some guidance on this topic permits of a better understanding of the logical structure of deduction. In particular the use of intuitive logical models leads to advances in abstract thought. To illustrate this is the purpose of the present article.

The Logic of Sets and Propositions

One of the most basic ideas of logic and mathematics is the idea of a class of objects or set of elements, that is to say of a collection of individual things regarded as a whole.

The 'extent' of a property is the set of elements having that property. For example, the property of an integer of being divisible only by itself and unity has, by definition, the set of prime numbers for its extent. The relation, between two straight lines, of parallelism has as its extent the set of all pairs of parallel lines.

The geometrical representation of a set by a simple closed curve and its elements by the interior points was made use of by Leibniz and Euler,[2] both of whom chose circles for their curves. With the aid of such diagrams one can illustrate relations between and operations on sets and thus make clear the corresponding relations between and operations on the properties of which those sets are the extents.[3] For example, if property A entails (or implies) property B, then the extent of A, E(A) is contained in (or included in) E(B), the extent of B.

To implication $A \Rightarrow B$	corresponds	inclusion $E(A) \subset E(B)$
To the equivalence of two properties $A \Leftrightarrow B$	corresponds	the identity of their extents $E(A) = E(B)$
To the conjunction of two properties (A and B) $A \wedge B$	corresponds	The intersection of their extents $E(A) \cap E(B)$
To the inclusive disjunction of two properties $A \vee B$	corresponds	The union of their extents $E(A) \cup E(B)$

The representation of sets of numbers by regions of a plane referred to cartesian axes is commonly used in the solution of systems of inequalities in two variables.

Another representation of sets is due to the mathematician Lambert, a pupil of Leibniz. It is used in discussing systems of inequalities in one variable. In it sets are represented by intervals, rays, or segments along parallel lines. According to whether the sets have a common part or not their representative intervals, rays or segments overlap or not. Such a diagram shows whether or not the inequalities are compatible, and if they are it enables one to see if one of the inequalities implies another or is equivalent to it.

These examples show how algebraic relations are studied by the aid of geometrical representations of sets of numbers. We have only to give such diagrams a wider logical meaning to obtain intuitive models of sets and of properties.

Propositional Logic

The calculus of propositions, i.e. of statements that are either true or false, is the first chapter of modern logic. It, too, can be illustrated by circle diagrams (see below). Another simple method is that of logical matrices. These we are now going to consider briefly.

1. In two-valued logic a proposition A can, with respect to a given reference system,[4] only be either true or false. To truth we make correspond the sign 1 and to falsity the sign 0.

The negation of a proposition A is the proposition that is false whenever A is true and true whenever A is false. We write it $\sim$A and read it not–A. We make a table of the values taken by A and $\sim$A thus:

A	$\sim$A
1	0
0	1

If we let "a" represent the truth value of the proposition A and "a'" that of the proposition $\sim$A, we have the arithmetical relation

$$a' = 1 - a$$

To restrict the values of a and a' to 1 and 0 it is enough to consider this equality as a congruence modulo 2.

2. As two propositions A and B can each be either true or false, there are at most four possibilities for the couple AB, namely: A and B both true, A true and B false, A false and B true, A and B both false. We represent these four cases by 11, 10, 01, 00 respectively.

Provided that the result of a logical operation on two propositions has a truth value depending only on those of the component propositions, the operation may be defined by stating, for each of these four cases, whether the truth value of the result is 1 or 0.

3. The logical equivalence of two propositions A and B, that is to say the proposition A $\Leftrightarrow$ B, which is true when the two propositions are either both true or both false, and which is false when A and B have opposite logical values, is defined by the matrix

A	$\Leftrightarrow$	B
1	1	1
1	0	0
0	0	1
0	1	0

In this table the four lines correspond to the four possible cases of truth values of A and B. Under each of the signs A, B, $\Leftrightarrow$, we have set down the value taken in each case by the propositions A, B, $\Leftrightarrow$.

If "a" and "b" are the values of A and B, to the equivalence A$\Leftrightarrow$B there corresponds the numerical equality $a = b$ (mod. 2).

The proposition $\sim(\sim$A), obtained by twice negating the proposition A, has the same truth value as A. The equivalence A $\Leftrightarrow$ $\sim(\sim$A), is, therefore, true no matter what the value of the proposition A. This is a law of logic, called "the principle of double negation".

Speaking mathematically we may say that the operation of negation is involutive, since when it is applied twice to a proposition the proposition is unchanged.

To the principle of double negation corresponds the numerical equality $a = 1 - (1 - a)$ (modulo 2).

4. The exclusive disjunction of two propositions A and B is the proposition that is false when A and B have the same truth value and true when their truth

values differ. We write it as A $\veebar$ B. It is defined by the matrix

A	$\veebar$	B
1	0	1
1	1	0
0	1	1
0	0	0

Observe that with addition modulo 2 we have

$$\begin{array}{ccccc} 1 & + & 1 & = & 0 \\ 1 & + & 0 & = & 1 \\ 0 & + & 1 & = & 1 \\ 0 & + & 0 & = & 0 \end{array}$$

so that the exclusive disjunction of A and B, A $\veebar$ B has as its truth value $a + b$ (mod. 2) the sum of the truth values of A and B.

5. Comparison of the matrices of A $\Leftrightarrow$ B and A $\veebar$ B shows that in each of the four cases these two propositions have opposite truth values. It follows that each of these composite propositions is equivalent to the negation of the other, and that their exclusive disjunction is true whatever the truth values of A and B. This is a law of logic written as

$$\begin{array}{rl} & (A \Leftrightarrow B) \Leftrightarrow \sim(A \veebar B) \\ \text{or} & (A \Leftrightarrow B) \veebar (A \veebar B) \end{array}$$

The laws of logic

$$\begin{array}{rl} & (A \Leftrightarrow \sim B) \Leftrightarrow (A \veebar B) \\ \text{and} & (A \Leftrightarrow B) \Leftrightarrow (A \veebar \sim B) \end{array}$$

are immediate consequences.

These examples have no other aim than that of illustrating the method of definition by logical matrices. This is much appreciated by pupils, not only on account of its clarity, but also because it gives a simple introduction to logical laws.

6. Provided the truth or falsity of the result of any logical operation on two propositions depends only on the truth or falsity of the two propositions, that operation can be defined by a logical matrix. We shall give some examples of this later. At the moment we wish to point out that the number of such operations having non-equivalent results can be determined. For, to define any one of them we have only to give the appropriate value, 1 or 0, to each of the lines of its matrix. This attribution can be made in only 16 different ways since the number of arrangements of two things taken four at a time with repetitions is 2^4 or 16.

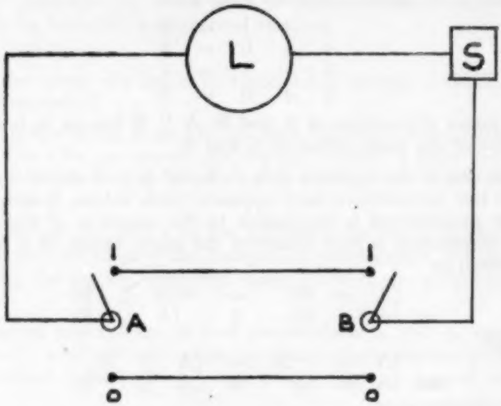
Electrical Models

A physical appliance can represent a proposition capable of being true or false if at any given time it must be in one or other of two mutually exclusive states. This is the case with a signal which is either at danger or at all clear, a lamp which is either alight or out, and a two-throw switch.

Provided the result of a logical operation on two propositions has a result depending only on their truth values, a model of the operation requires only that the device representing the result be suitably actuated by the devices representing the component propositions.

These requirements are met by an electrical circuit comprising a source S of electricity, a lamp L and two switches A and B whose moving parts make contact with either one of two terminals marked 1 and 0. [5]

1. Consider the lighting up of a lamp in the following circuit containing two single-pole double-throw switches.



The lamp lights up only for 10 and 01, in which the moving parts of A and B are differently placed. The arrangement is a model of the exclusive disjunction

A	$\underline{\vee}$	B
1	0	1
1	1	0
0	1	1
0	0	0

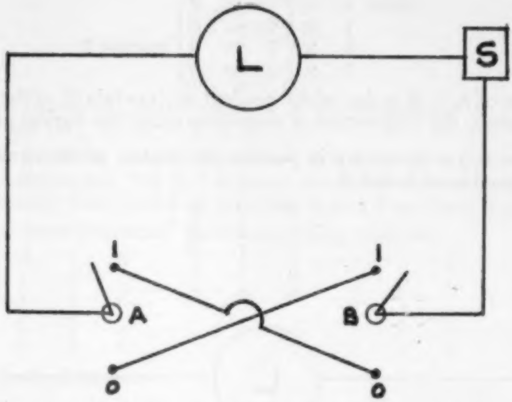
or of addition modulo 2.

As there are only two positions for each of the switches A and B, 1 or 0, there are just four cases for the couple AB: 11, 10, 01, 00. The lamp lights up for 11 and 00 only, in which the moving parts of A and B are in the same positions. This circuit is therefore a model of the equivalence A $\Leftrightarrow$ B, whose matrix is

A	$\Leftrightarrow$	B
1	1	1
1	0	0
0	0	1
0	1	0

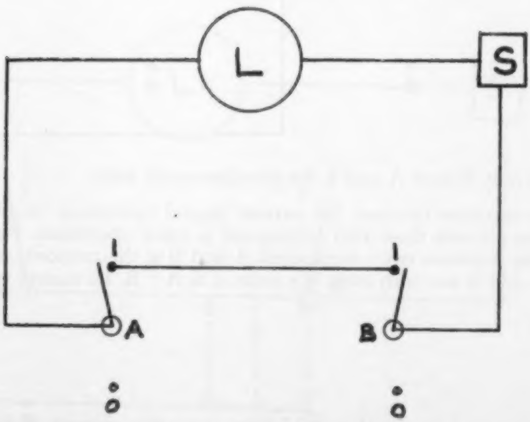
or of the equality $a = b$ (modulo 2).

2. Let us cross the wires joining the terminals of A to those of B. We obtain the following circuit.



3. A circuit with the lamp, switches, and source in series has as its matrix that of the conjunction

A	$\wedge$	B
1	1	1
1	0	0
0	0	1
0	0	0



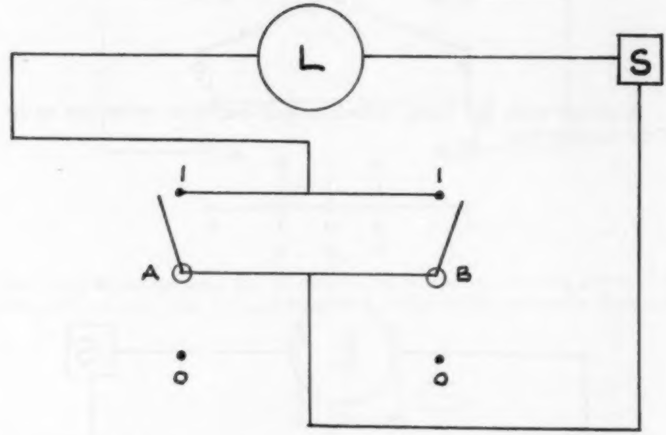
which is only true if both A and B are simultaneously true.

$$\text{Since} \left.\begin{array}{l} 1 \times 1 = 1 \\ 1 \times 0 = 0 \\ 0 \times 1 = 0 \\ 0 \times 0 = 0 \end{array}\right\} \text{modulo 2}$$

the truth value of A $\wedge$ B is that of the product ab (modulo 2) of the values of a and b. For this reason, the conjunction is sometimes called the logical product.

4. If the two switches are in parallel the matrix of the circuit is that of the inclusive disjunction of A and B

A	$\vee$	B
1	1	1
1	1	0
0	1	1
0	0	0



which is false only if both A and B are simultaneously false.

5. The relations between the various logical operations enable us to derive from these four circuits those that correspond to other operations. For example, the incompatibility between two propositions A and B is the proposition which is false only when A and B are both true. We write it as A | B. Its matrix is

A	|	B
1	0	1
1	1	0
0	1	1
0	1	0

This shows that its logical value is in every case the opposite of that of the conjunction

A	$\wedge$	B
1	1	1
1	0	0
0	0	1
0	0	0

A | B is thus equivalent to the negation of the conjunction A $\wedge$ B. Thus we have the law of logic (A | B) $\Leftrightarrow$ ~(A $\wedge$ B).

On the other hand the cases in which A | B is true, 10, 01, 00, are precisely those in which the disjunction ~A $\vee$ ~B is true, since this is false only when ~A and ~B are simultaneously false, that is to say when A and B are both true.

Moreover the consideration of the corresponding matrices

A	|	B
1	0	1
1	1	0
0	1	1
0	1	0

~A	$\vee$	~B
0	0	0
0	1	1
1	1	0
1	1	1

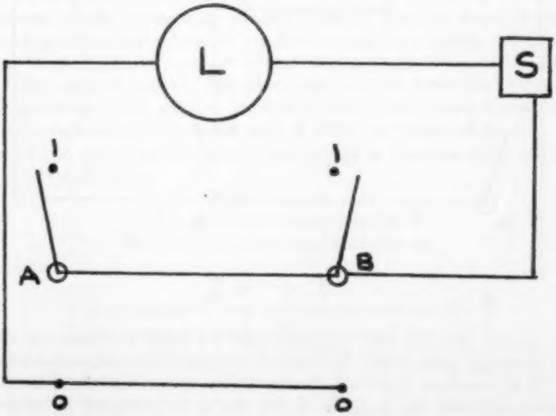
makes plain the truth of the law

$$(A \mid B) \Leftrightarrow (\sim A \vee \sim B)$$

Thus the propositions ~(A $\wedge$ B) and (~A $\vee$ ~B) being each equivalent to the inclusive conjunction of their negations, is due to De Morgan. It has as corollary a second law giving a negation of an inclusive disjunction:

$$\sim(A \vee B) \Leftrightarrow (\sim A \wedge \sim B)$$

Since the incompatibility A | B is equivalent to the disjunction ~A $\vee$ ~B, the circuit corresponding to A | B is obtained from that of the inclusive disjunction A $\vee$ B by interchanging the roles of 1 and 0, since this is equivalent to replacing A by ~A and B by ~B. We thus have, for incompatibility



6. The material implication of proposition B by proposition A is the proposition that is false for A true and B false and true in the other three cases. The matrix is thus

A	$\Rightarrow$	B
1	1	1
1	0	0
0	1	1
0	1	0

$A \Rightarrow B$ is false only in the case 10, the only one in which the conjunction $A \wedge \sim B$ is true. We thus arrive at the law

$$(A \Rightarrow B) \Leftrightarrow \sim(A \wedge \sim B)$$

By De Morgan's first law we have

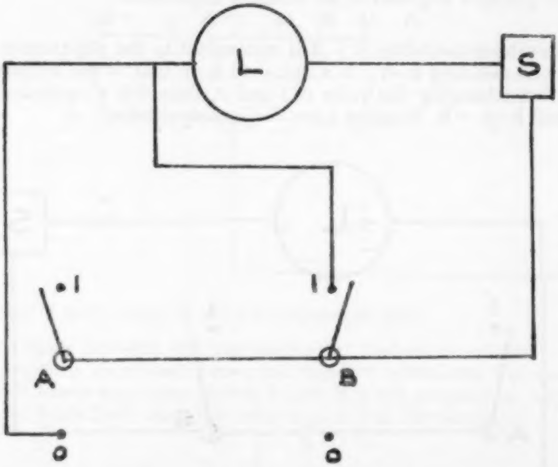
$$\sim(A \wedge \sim B) \Leftrightarrow (\sim A \vee \sim\sim B)$$

and by the principle of double negation

$$(\sim A \vee \sim\sim B) \Leftrightarrow (\sim A \vee B)$$

Thus $(A \Rightarrow B) \Leftrightarrow (\sim A \vee B)$

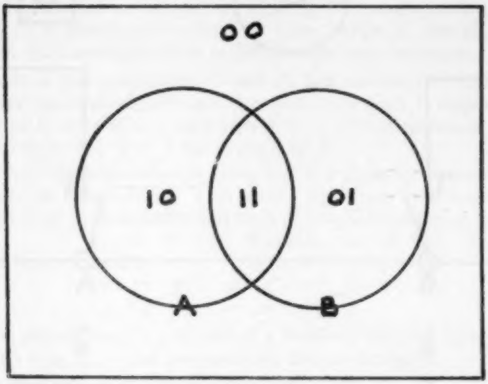
To obtain the circuit corresponding to $A \Rightarrow B$, we have only to replace A by $\sim A$ in that of inclusive disjunction, i.e., to interchange 1 and 0 for A. We thus arrive at



7. The above circuits and matrices can be used to illustrate operations on, and relations between, properties. We take 1 to mean the possession and 0 the absence of a property. Furthermore there is a correspondence between the four cases presented by two propositions and the regions of the plane determined by two Euler circles representing the extents of two properties A and B.

To each of the four cases there corresponds a region characterised by the conjunction of two properties

$$\begin{array}{lcl} 11 & : & A \text{ and } B \\ 10 & : & A \text{ and } \sim B \\ 01 & : & \sim A \text{ and } B \\ 00 & : & \sim A \text{ and } \sim B \end{array}$$



It follows that number of electrical circuits consisting of a source, a lamp, and two single-pole, double-throw switches, and not equivalent to each other from the point of view of the lamp lighting up is equal to the number of different maps obtainable by colouring the above diagram with at most two different colours. This number is the same as that of the non-equivalent matrices, namely sixteen.

The purpose of the foregoing remarks was to furnish some simple examples of models having the same abstract structure and to explain some useful logical ideas. In class these explanations are given at convenient moments during the last three years of the science course. The electrical circuits described were all set up by the pupils themselves after group discussion and individual research during the period between examinations and the end of their last term at school.

The aim of the work was to oblige the pupils to become fully aware of the two essential ideas of implication

$$A \Rightarrow B$$

A is a sufficient condition for B
B is a necessary condition for A,

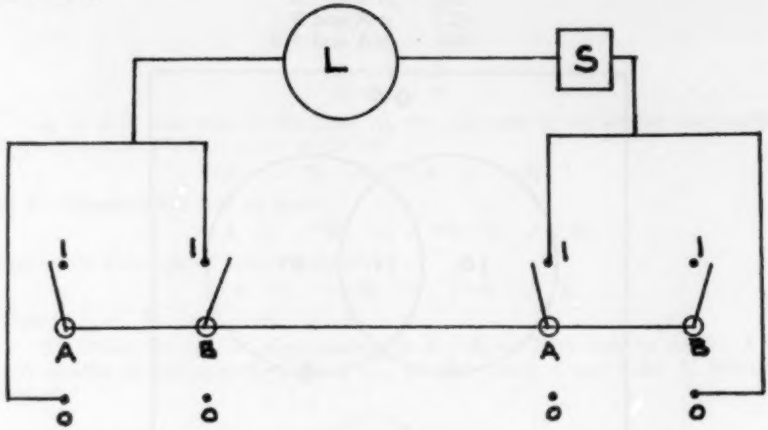
and of equivalence

$$A \Leftrightarrow B$$

A is a necessary and sufficient condition for B
B is a necessary and sufficient condition for A.

In mathematics the equivalence $A \Leftrightarrow B$ is often established by the proof of two simultaneous reciprocal implications $A \Rightarrow B$ and $B \Rightarrow A$. This method is based on the fact that the conjunction $(A \Rightarrow B) \wedge (B \Rightarrow A)$ is logically equivalent to the equivalence $A \Leftrightarrow B$, as the double-headed arrow indicates.

If we, therefore, connect in series two arrangements of switches corresponding to mutual implication, we obtain a circuit in which the lamp lights up in cases 11 and 00 only.



Boolean Algebra

The logic of classes (or sets) and the logic of propositions have rules of calculation that make of them algebras of a particular kind, called Boolean algebras, after the name of the English mathematician, George Boole (1815-1864).

Boole was one of the first to conceive that algebra need not concern itself exclusively with numbers and relations between them, but might be considered as a formal calculus applicable to non-numerical elements indeterminate in nature and subject only to operational relations.

Several axiomatic characteristics of Boolean algebras are possible. We shall con ent ourselves with stating some properties common to all Boolean algebras and from which a system of postulates may be chosen.

A Boolean algebra is a formal system consisting of a non-empty set K of elements $x, y, z, \ldots$ of indeterminate nature, the set K being given a structure by :—

1. Relations.

   (a) a relation of equivalence, written $x = y$, which is reflexive, symmetrical, and transitive. Any two elements x and y of K satisfy either the equivalence $x = y$ or its negation $x \neq y$. There are at least two non-equivalent elements of K.

   (b) a relation of order, written $x \subset y$, which is reflexive and transitive, and in addition anti-symmetrical, i.e. for every x and y of K, if $x \subset y$ and $y \subset x$, then $x = y$

   These three properties are satisfied by the relation $\leq$ between real numbers.

2. **Operations.**

(a) two binary operators, $\cup$ and $\cap$, that to every pair of elements x and y of K assign a third element of K, defined to within an equivalence. Each of these binary operations is commutative and associative like ordinary addition and multiplication, and also idempotent, that is to say for every x of K we have $x \cup x = x$ and $x \cap x = x$, which is not the case with addition and multiplication in elementary algebra. Moreover, each of the operations $\cup$ and $\cap$ is distributive over the other, while in the algebra of complex numbers only multiplication is distributive over addition.

Each of the operations $\cup$ and $\cap$ has an identity element determined to within an equivalence and denoted by 0 and 1 respectively. Thus for every x of K, $x \cup 0 = x$ and $x \cap 1 = x$. The elements 0 and 1 satisfy the order relation $0 \subset x \subset 1$ for every x of K.

(b) a unary operation that to every x of K assigns an element x' of K, defined to within an equivalence. This unary operation is involutery, i.e. $x'' = x$ for every x of K. It satisfies the laws of complementation.

$$x \cap x' = 0 \text{ and } x \cup x' = 1$$

and the laws of duality.

$$(x \cap y)' = x' \cup y'$$
$$(x \cup y)' = x' \cap y'$$

The logic of propositions is a model of a Boolean algebra. One has only to consider the elements $x, y, z, \ldots$ as propositions and to interpret

1. (a) the relation $=$ as logical equivalence $\Leftrightarrow$

   (b) the relation $\subset$ as material implication $\Rightarrow$

2. (a) the operations $\cup$ and $\cap$ as inclusive disjunction $\vee$ and conjunction $\wedge$, respectively. The elements 0 and 1 are an arbitrary false proposition and an arbitrary true proposition, respectively. (All true propositions are equivalent, and so are all false propositions)

   (b) the complement of a proposition as its negation.

   The three laws of complementation then express the principle of double negation,
   the principle of non-contradiction and that of the excluded middle,
   the laws of De Morgan.

The logic of classes (the algebra of sets) is another model of Boolean algebra. The elements are the sub-sets of a given set C. We interpret

1. (a) the relation $=$ as identity

   (b) the relation $\subset$ as inclusion.

2. (a) The operations $\cup$ and $\cap$ as union and intersection respectively, 0 as the empty set and 1 as the universal set C.

   (b) the complement of a set x with respect to C is the set of elements of C not belonging to x.

   The three laws of complementation then state that the complement of the complement of x with respect to C is the set x itself, the intersection of a set and its complement is empty, their union is the set C. The complement of an intersection is the union of the complements; the complement of a union is the intersection of the complements.

The calculus of 1 and 0 in the truth matrices, and the calculus of truth values are both examples of Boolean algebra in which the set K consists of only two (non-equivalent) elements.

These examples are abstract models of structures still more abstract. This fact enables us to understand why Bertrand Russell had some reason to affirm that "Pure mathematics was discovered by Boole in a book that he entitled 'The Laws of Thought'."[6]

References:

1. G. Boole, *Mathematical Analysis of Logic*, 1847; *The Laws of Thought*, 1854. A. De Morgan, *Formal Logic*, 1847.
2. Euler, *Lettres a une princess d'Allemagne.*
3. W. Servais, *Equations et lieux geometriques. Synthese Logique. Mathematica et Paedagogia*, no. 5, pp 6 à 19.
4. W. Servais, loc. cit.
5. W. Servais, *Modeles, objets concrets et symboles. Mathematica et Paedogogia*, No. 8. p 33 onwards.
6. Russell, *Mathematics and the Metaphysicians*, in *Mysticism and Logic.*

SECONDARY MODERN SCHOOL MATHEMATICS - III
THE YEAR OF IMPORTANCE

H. FLETCHER

Undoubtedly the first term of a pupil's life within the Secondary Modern School is the deciding period for future enthusiasm and enjoyment of mathematics. It begins the year which should prove to many frustrated pupils that mathematics is something which can be enjoyed — how important this is and how important this first year is!

Far too many arrive with unhappy recollections of arithmetic—it was a subject in a selection examination and whereas they felt reasonably happy about the mechanical examples, those problem sums caused havoc—in fact arithmetic was the cause of their failure. I have heard this argument so many times that teachers *must* make certain these pupils do not receive the type of work which makes them say, "This is where we came in".

In this first important year, the teacher's vision is the guiding light to success or failure. Arithmetic must give way to mathematics; constructive experiment and discussion blots out computation tricks (though some mathematical grammar is always essential). Plain facts fail to survive any period of time and the creativeness they inspire is of little depth. I am aware certain facts must be known but they should

lead to purposeful and constructive activity. The teacher must spot the moment when a concept is fully appreciated and the next step ready for introduction, remembering that this readiness is not determined by chronological age.

I cannot overstress the importance of discussion during this first year. From discussion the teacher can appreciate how much mathematical thinking is taking place, how much of the teaching has value. I am quite prepared to spend considerable time each lesson discussing some mathematical concept—there may be little written work in the exercise book but do we judge the success of our work by this book? It may be argued that too much discussion will delay the progress of the scheme but is it vital that I must have taught decimals by Christmas? It is a grave situation if a scheme has been arranged which necessitates reaching a certain concept in a given time. It is experiences which have become part of the pupil and the mathematical thinking he gains from them which can decide how far the scheme will stretch—and there is no limit.

How can we introduce this year of importance? Please let us avoid a grading examination in the first week. The children's scores must be unrelated to the child's standard; he has just finished a long holiday and needs some time for readjustment and recovery.

I prefer to take all the entrants myself, in the school hall, for the first weeks. Eventually, I hope to divide the intake (140 pupils) into five groups and the masters who will be assigned to these groups will be in attendance, helping and watching my presentation.

An introduction of historical trend will give a welcome change. A short series of "In Town Tonight" where great mathematicians are interviewed and given examples of what we owe to their conclusions can prove very exciting. A keen master will not object to being Archimedes, at least for the purposes of this interview.

I believe in holding an exhibition of measurement. Every example of measurement I can find is included in this exhibition and groups of new pupils are 'let loose' to explore and to ask questions. Discussion soon develops and good oral English is produced. The woodwork and metalwork masters have provided much for this exhibition, examples on show deal with the measurement of length, weight, money, time, capacity, angles, temperature, etc. The old fashioned eggtimers arouse great interest, so do the anglemeters and theodolites. After their initial visit to the exhibition (held in the maths-room) group discussions take place in the hall. The boys write down their own comments such as: 'a pint is a measurement (the best known in my area)'; '5 cu.ft. is a measurement'; 'A coat can be measured in money'; 'placing of legs on a table is angle measurement'; 'calories give an energy measurement'; 'My journey to Paddington could be a time measurement'; 'the length of the stride of a guardsman is a time measurement prior to the coronation march'; 'a chisel is a measure for joint making'; 'speech, manners and appearance measure manliness'.

These are samples of what boys have written after discussion. The following morning you can expect a further batch of correspondence dealing with measurement. Many statements will be incorrect but these can be used in discussion to encourage deeper thought. The exhibition will lead to discussion on angles and I have

found meccano sets soon arrive and assist in introducing a new vocabulary—angles of direction, revolutions, right angles, vertical angles, horizontal, obtuse, acute, corresponding and alternate. It is here that paper folding can be of great value if not over emphasised, while envelope stitching can give new life to the boy who felt he was a failure. Shapes are discovered, they are noted again by the boy on his way home, for his garden gate has given him a clue on angles and the triangle. Two weeks may pass by but much has been done outside school hours and enthusiasm has grown.

Fun with numbers begins and a great effort should be made here. Laws can be found by these young detectives:

0_1 (1st ordinary number) $= 1$
0_2 (2nd ,, ,,) $= 2$
....
0_n $= n$

E_1 (1st even number) $= 2$
E_2 $= 4 = 2 \times 2$
....
E_n $= 2 \times n \; (= 2n)$

D_1 (1st odd number) $= 1 \; (= 2 - 1)$
D_2 $= 3 \; (= 4 - 1)$
....
D_n $= 2n - 1$

Similar "codes" can be found by pattern practice using triangles and squares or small circles. Boys enjoy finding out laws and Algebra need not be mentioned. The fun of X-raying numbers can be useful and amusing. Let us find out all we can about 60, 12, 100 and we add the word factor to our vocabulary.

The finding of patterns and their laws can be exciting not only for the 'A' but also for the 'B' and 'C' streams. I have found the graphing of tables to be of great value especially with the 'D' stream.

Boys and girls enjoy giving meaning and language to number, money, weight, time and length. With the chess board, they can discover $(1 + 2 + 3 + 4)^2 = 1^3 + 2^3 + 3^3 + 4^3$ and I have found they bring in their parents in their thirst for discovery. Letter distribution in a newspaper and its connection with a 'John Bull' printing set and a study of the morse code will start a useful discussion. Another idea which has received special notice is asking questions about a set of statistics. e.g.

Cinema Takings

M.	£10
T.	£8
W.	£25
T.	£7
F.	£30
S.	£45
S.	—

i) Would this cinema be situated in a village, town or city?
ii) What was the bye-law in this area?
iii) When was half day closing?
iv) Why £45 on a Saturday?

Varied answers will be given and all could be correct if they were fully qualified. I was given an answer—"Tuesday is half day closing because only a few people went to the cinema and it was a saint's day". "The takings on Thursday were low because it was the local W.I. trip to Rhyl". Discuss every answer and encourage thinking, then graph the takings (line and bar graphs, histograms).

Why not discuss the meaning of money, etc? What does £1 mean to you?—to your mother? Does it mean 8 × 2/6? On asking a boy what £1 would mean to his mother I received the answer "Two days' cigarettes", a great lead to further discussion.

It is a useful activity to take a sum of money mentioned in a newspaper article, extract and simplify the amount and return your simplified version to the original context, then comment. One boy noted '£3,000,000 for the manufacture of Comets'. He produced a report stating this money would amount to 1/2½d per head of population and commented, "Is it worth 1/2½d per head for more comfortable, faster air travel? Is it worth 1/2½d per head for Britain to be a highly rated country in air power? To me the answer is yes". (He was an 'A' boy).

Thus in the first two months we have had fun with numbers, money, length and the general concepts. We have introduced geometry, used compasses and set squares and Algebra has been introduced by finding laws based on number. I always try to base this early work on pattern and movement within the pattern, this movement being very important when discovering laws. Discussion on real things leads to orderly results and is followed by abstract presentation. I must use the knowledge gained to strengthen tables weaknesses and to continue to excite enthusiasm. This I do by introducing thoughts on one, two and three dimensions. Area needs a very full discussion followed by experimentation. I develop it this way:

(a) easy numbers are used

(b) the answers are written in full English e.g. 12 sq. in.—'I require the equivalent of 12 squares, each side one inch, to cover the rectangle'.

(c) Area of square after geometrical discussion (angles, parallel sides, diagonals are known words already)

(d) Area of rectangle, parallelogram, tiles required, borders, walls of a room. Practical work is carried out in every case and easy numbers help the tables revision.

(e) If the perimeter of a rectangle is 36 in. find the area when the breadth is 1 inch, 2 inches, 3 inches, etc. Graph the results and give observations.

(f) The triangle and the trapezium. Is the triangle a strong figure? Compare it with the rectangle—making both with a meccano set.

Now volume is discussed and experiments carried out leading on to the cuboid and triangular prism. Geometry and algebra are always coming to the fore—the teacher's skill is the key here and ideas are inculcated only from the background of the teacher's own knowledge. All this work will take a term but its value is evident later. Enthusiasm is aroused and revision is accomplished by putting the ideas into new contexts.

Now is the time to discuss "When do we add, subtract, multiply, divide—in every day life?" Lists are made, examples set, answers estimated and proved. It is good to ask the child whether or not he considers the examples sensible. Averages can serve as practice in addition and division and lead to interesting discoveries as to their relevance and dangers. Eventually it will cause amusement if the following is given for comment

$$\begin{array}{r} 3\,7\\ 2\,7\\ \hline 6\,0\\ \hline\hline \end{array}\qquad \begin{array}{r} 3\,7\\ 2\,7\\ \hline 6\,2\\ \hline\hline \end{array}\qquad \begin{array}{r} 3\,7\\ 2\,7\\ \hline 5\,1\,4\\ \hline\hline \end{array}\qquad \begin{array}{r} 3\,7\\ 2\,7\\ \hline 6\,6\\ \hline\hline \end{array}\qquad \begin{array}{r} 3\,7\\ 2\,7\\ \hline 6\,4\\ \hline\hline \end{array}$$

All solutions are correct

By the end of the year, fractions and geometry are close friends, fractions have been introduced to the area of shapes already known and to solids. The circle has introduced the concept of limits: practical work has led to the re-arranging of a circle into a rectangular shape and a solid circle of string into a triangle. From these new shapes the area can be determined and discussed. I do suggest that solutions of area and volume examples are considered approximations. Of this I am certain, if the early "pattern-and-law" practice with number has been thoroughly discussed, no difficulties will arise in connection with a formula. With fractions, I find boys and girls enjoy studying

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{2.3}$$

$$\frac{1}{3} - \frac{1}{4} = \frac{1}{3.4}$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{1}{5} - \frac{1}{6} = \frac{1}{5.6}$$

Hence, $\dfrac{1}{2} - \dfrac{1}{6} = \dfrac{1}{2.3} + \ldots\ldots + \dfrac{1}{5.6} = \dfrac{1}{3}$

Ratio has made its introduction earlier in the plan, fractions develop the concept and percentages carry it further. Is not Simple Interest ratio thinking? The common fractions, decimals and percentages can be taken together.

The Year can thus be made one of exciting experiment, the class will carry out many further experiments at home and in my experience demand and take part in mathematical club activities after school. Thinking has become the order of the day and this thinking about mathematics spreads to other subjects. It is highly probable that the class will become 'semi-sceptical' but there is no reason why they should always accept what they are told.

No specific text-book is used; the teacher makes his own book using many publications in his search for examples showing sequence and interest, choosing those examples which suit the children's interests, aptitudes, attitudes and attainments.

IS SIXTH FORM TEACHING GOOD ENOUGH?

The teaching of mathematics has been the subject of much discussion in the National Press in recent months. Teachers of mathematics have been under a heavy fire of criticism and advice, but, as yet, only a few stray shots have penetrated the Sixth Form blockhouses. Perhaps this is because these contain the best pupils, most of the better-qualified teachers, the smallest classes: have the best working conditions and the advantages of a clearly defined aim. There is a marked complacency among teachers fortunate enough to be taking Sixth Forms—at least, as regards their own share of the classroom work. Is it justified?

What criteria can be used to judge the quality of the teaching of mathematics in the Sixth Form? The most objective standard is, no doubt, performance in the Advanced Level of the G.C.E. But success in an examination does not necessarily measure anything more than the ability to be successful in an examination. It gives only superficial information about enthusiasm, intellectual curiosity and depth of understanding, which may well be considered important products of successful teaching. In spite of the dangers of subjective assessment, it cannot always be avoided. The views of University teachers, who have to take over students from the Sixth Forms, provide a necessary corrective to the defensive over-optimism of school-eachers.

In the early part of 1959, a questionnaire* was sent to the Heads of the Mathematics Departments of English Universities. This invited them to state their opinions of the adequacy of the mathematical training which had been received by their incoming students. More than a dozen replies were obtained from Universities and University Colleges, and, even assuming that all of those who did not choose to reply were satisfied with the existing situation, enough evidence remains to give teachers occasion for much thought.

Here are some extracts from the replies to the question: "Are you satisfied with the training and mathematical background of students entering your department to read Honours in Mathematics or Mathematics as an ancillary subject? If not, what particular shortcomings do you find in attitude to the subject or in basic knowledge?"

> "It is rare to find a first year student who looks on mathematics as anything other than a collection of empirical skills and tricks. There is no grasp of the need for proof, and constant failure over the most elementary logical principles. Few students understand that mathematics needs serious study and cannot be picked up by working a few exercises".

> "Pupils at school are not required to understand what they are supposed to have learnt, and the modern tendency in the teaching of mathematics is to avoid the use of the brain as much as possible, whereas the only justification for the teaching of mathematics is mental training".

> "The main failing of pupils is a certain lack of enterprise in their outlook. They seem to have been so effectively drilled at school that it comes as quite a shock when they find they have to do anything for themselves".

> "The emphasis in schools is on preparation for the problems that pupils meet in their examinations. There is often a lack of understanding of the prin-

* Prepared by the London Study Group for the XIIIth Conference of the International Commission for the Study and Improvement of the Teaching of Mathematics.

ciples which these problems are meant to illustrate. Also, a pupil coming straight from school seems frequently unaware that mathematics has progressed since the time of Newton".

"Students are usually very well trained—that is, drilled in technique—but lack a general background, never having been encouraged to read the history of mathematics or any general books".

"There is a general weakness in the understanding of the fundamental principles of the subject. Many of the students, in fact, just do not know what the subject is, or what they are trying to do with it. They have been pushed at too rapid a rate through a greatly overloaded and inflated syllabus, and have not had the time to assimilate the basic ideas".

"There seems to be less interest in the subject for its own sake than one would hope for. There is a tendency to cover too wide a range and to encourage technical manipulation rather than to instil an interest and respect in the general principles of mathematics".

"Honours students from the schools have been well-trained in techniques and the basic knowledge of the traditional school syllabus. However, they frequently lack depth of understanding of what they have learned. The same faults are noticeable in those taking mathematics as an ancillary subject".

"Students who come to us have little or no idea of why they study mathematics in school. The points new to them are the insistence on detailed mathematical logic in Pure Mathematics, and an understanding of the physical principles in Applied Mathematics. Many schools have pushed school studies to quite unnecessary depths of specialisation. Manipulative skills in pure mathematics are lacking, as well as an understanding of the basic principles of theoretical mechanics".

"The main criticism is that pupils are taught a bunch of techniques without an understanding of the underlying principles. Students miss seeing the unity of mathematics and a good deal of its value and purpose".

There is an overwhelming unanimity of complaint here. Even allowing for the fact that some of the writers may have given in to the temptation to exaggerate their difficulties, there is no reasonable doubt that a large proportion of the students lack an intelligent understanding of the principles of the subject. They are not in love with mathematics. *And these students are the schools' best pupils.*

What is the use of pointing to the number of Advanced Level passes as an indication of successful teaching? Since all of these students are securely in possession of the appropriate admission qualifications, that criterion appears to be of very dubious significance. Probably schoolteachers, who should certainly know better, are guilty of supporting the fiction that success at A.L. is all that matters. Honesty must force them to admit, however, that this brand of success may conceal frightening shortcomings.

Having once faced the fact that all is not well, it is necessary to consider possible remedies for the situation. The following comments were received in answer to the question: "In what way do you consider the schools could improve the preparation of their pupils for a University course?"

"One can only suggest that more understanding or inspired teaching might start off the students with a livelier interest".

"It would be a good thing if pupils could be encouraged to read books for themselves".

"The student should have been taught at school with more regard for accuracy in Algebra and Arithmetic. They should place less value on lucky guesses and more on systematic methods. The general idea of trial and error in mathematics should be expounded. The schoolmaster should impress upon his pupils that a correct and complete solution of one problem is more valuable than incomplete solutions to three problems".

"School teaching should be less specialised, with more concentration on stimulating the imagination. After two years in the Sixth Form, the student should move on to a minimum of four years University training. In addition, there should be a proper collaboration between schools and Universities to mark out the function of each and arrange that University courses follow rather than overlap those of the schools".

"The present school syllabuses do not contain any work associated with what may be termed 'modern' mathematics, and we hope that the schools may be able to introduce some of the work mentioned on modern topics at an elementary level. At all times we should like to see a greater depth of understanding of the techniques employed. More frequent contacts between school and University teachers would help a great deal. Some reorientation of examination questions is also desirable to avoid the high marks obtainable on occasions by pupils carrying out routine operations merely by rote".

"Schools could improve their pupils by teaching them a wider syllabus. It would be impossible for these schools to devote so much time to mere routine work".

"Part of the trouble is that there are insufficient places in the Universities, with the result that school teachers often concentrate on preparing their pupils for University tests to the exclusion of those wider aspects of mathematics which are so important to anyone who proposes to spend some of the best years of his life as a student of the subject".

"Part of the solution is for the Universities to remodel the examinations for which they are responsible. Provided the reforms are realistic about the capabilities of the pupils, then resistance to this should be overcome".

"There is an immediate need for a complete examination of the G.C.E. Advanced Level".

"I doubt if the schools can do much in the absence of a radical change in the examinations. I would like to see the schools doing really careful work on a much smaller and more elementary syllabus".

There isn't the same agreement about the cure as there is about the diagnosis. Not that this is surprising. What would the teachers in the schools say? When they had laid the blame (supposing they would accept that need for it existed) at the door of the G.C.E. examiners, or inadequacies of the teaching at earlier stages, or the poor calibre of the pupils, or whatever, would they feel that any small portion attached to themselves?

Obviously, there is a very great need for a much more detailed investigation of the scope and extent of the present problem and for a search, at all levels, for ways of improving the existing situation. It is to be hoped that examinations and syllabuses will be overhauled. But it must be remembered that reform does not always come from above. It comes, as it must, from the reformers, who, at any level, are conscious of the deficiences of the system within which they work, are impatient of mere expediency, and look again for the spirit behind the letter of their job.

D.H.W.

TRANSITION FROM SIXTH FORM TO UNIVERSITY INTERNATIONAL CONFERENCE REPORT

TREVOR FLETCHER

The International Commission for the Study and Improvement of the Teaching of Mathematics held its thirteenth meeting in Denmark during the middle of August. Over forty teachers from nine countries took part, and the main theme for discussion was the mutual responsibilities of schools and universities. The conference opened at the Stadtsgymnasium at Nyborg. This school displays the high qualities of contemporary Scandinavian architecture, and it was a felicitous omen for this forward-looking commission to meet in a school which is still being built.

As members of the group described the educational systems of their separate countries a bewildering variety was immediately apparent, but as discussion proceeded it became clear that they had certain problems in common, and that these problems were deep-seated and little would be gained merely by comparing the administrative expedients which had been adopted in the different countries.

Many European countries transfer pupils to High Schools at the age of 15–16; and the system works satisfactorily in spite of the strong *a priori* objections which are raised in England. This system should be studied carefully as it may make more effective use of the highly qualified specialist teachers who are in short supply everywhere. Teachers in Denmark teach for about twice as many hours per week as teachers in Spain—but they teach classes of about half the size. In many countries there is a nation-wide examination *set by the schools* which qualifies automatically for university entrance. In France the examination for baccalaureat imposes very high intellectual standards and far-reaching questions of social status are bound up with the examination system.

In Scandinavia the specialised technical colleges are able to select only the candidates with the highest qualifications and those passing with lower marks are left over for the Universities! In Spain state education touches only 25% of the population, and there has recently been intensive development of technical education in "work institutes" which provide an educational service which has hitherto been lacking in the country.

The system in Switzerland is impossible to understand unless one has lived with it. University qualifications in one canton may not necessarily permit one to practice the corresponding profession in another. Also individual communes value their independence very highly, and this can affect local education.

Examinations were discussed, and there was the usual concensus of opinion that they must be accepted as a necessary evil. But the style of mathematical question set in England is definitely towards one extreme, and it seems that our examiners might well experiment with questions designed to test understanding rather than the recognition of particular tricks. We might, for example, have much to learn from the type of question involving analysis of, and commentary on, a given piece of expository text. Examination authorities should also be more aware of the repercussions of their policies beyond the immediate sphere of the candidates actually taking the examination. In England the scholarship policies of Oxford and Cambridge result in many grammar school pupils undergoing highly specialised courses

of study even when they never attempt these scholarships. Also the G.C.E. examinations designed for pupils of a particular academic type greatly determine the syllabus of many less-gifted pupils who never take the actual examination. The authorities should accept responsibility for these secondary reactions.

The discussions were occasionally handicapped by there being a slightly inadequate representation of university teachers, and by there being few teachers present with direct experience of both fields. This alone showed the need for university teachers to take a greater interest in teaching at lower levels, and for methods to be developed whereby there is a greater exchange of ideas and of people between the two fields. The far closer integration of teacher-training with the university course which the Danes achieve with their five-year course of study for a degree may be something which we should investigate.

There are obvious differences in maturity between students in schools and universities, but throughout education the Commission felt a need for a much better understanding of the psychology of learning and of the formation of mathematical concepts. Mathematicians are understandably shy of venturing into these notoriously uncertain fields when immediately to hand they have a pure discipline in which the greatest possible degree of certainty may be reached, and it is the particular quality of the International Commission that its members are prepared to approach these almost untouched problems, and furthermore feel under obligation to do so even if on a short-term view their discussions seem wordy and futile. Discussions at national level nearly always see the problem only in terms of devising syllabuses and never in terms of basing methods on a fundamental philosophy. It is the feeling of the Commission that the lesser problems of content can only be solved against the wider background which results if one dares to ask questions about the whole purpose of education and the whole process of mathematical learning.

The second part of the conference was again in surroundings of the greatest architectural distinction. The new, ivy-covered buildings of Aarhus University, situated in a wooded, undulating park are one of the sights of Denmark. But with the change to a less intimate atmosphere a certain sense of direction was lost and the discussions on the main theme of the meeting had far less pace. Working by a 'sense of the meeting' rather than by a set agenda it became clear that different ideas were crystallising out.

This century has seen fundamental changes in what mathematicians consider to be the essential core of their subject—the 'mathematique de base' to use the French phrase employed at the conference. One manifestation of this far-reaching change of attitude and emphasis is the work of the Bourbaki movement in France.

This corporate body of mathematicians—virtually a secret society—has published over the years a series of text-books relaying the presentation of mathematics at university level on new foundations. These changes must be carried into the schools and must become part of our general culture. This task seems impossible if one fails to see that these changes are *not* increasing complications of old ways of thought which are already too difficult for many pupils; they are a thorough-going simplification, a complete re-orientation of the whole pattern which develops mathematics in a way which is not only better by its own internal standards, but which is also in far greater sympathy with the known experimental facts of psychology concerning the formation of mathematical concepts in the mind of the child.

When these facts are understood it need cause no surprise to find that the elements of algebra may be taught with success before the elements of arithmetic (which is a more complicated system), or that fifteen-year-old students at a Froebel training college in Belgium have been doing creative work in a part of topology which only a few decades ago was at the very frontiers of mathematical knowledge. The Commission is preparing a complete series of school texts inspired by these ideals, and their appearance is awaited with keen interest. England is being far slower than some other countries to study these new ideas, and in this field of mathematics teaching, in spite of the high technical standards achieved by our very best pupils, we are by no means a leading country. A generation from now we may find ourselves outdistanced by countries which seek new ideas with greater vitality.

By general agreement the next meeting of the International Commission will be on this contemporary 'mathematiques de base'; and it is hoped that the discussions will not only cover the view-points of pure mathematicians and educational psychologists, but also of those scientists and technologists who are applying the new mathematics to a wider range of human activities than ever before.

GEOMETRY IN THE PRIMARY SCHOOL.

JOHN LUNN

The immediate re-action of many to this title may be that geometry is no concern of the primary school and that this is one more instance of the over-burdening of the primary school curriculum to satisfy secondary school pressures and the ultimate needs of a society that finds itself short of scientists, technicians and mathematicians. A little thought may serve to convince the reader otherwise, and show what aspects of the subject come within the province of the primary school.

Far too many teachers and adults think of geometry in terms of their own secondary school experiences. For them it was a strange subject that had to do with lines, figures, angles and circles and involved lots of proofs of equality which had little meaning or purpose. When they wrote 'Q.E.D.' at the end of their last geometry problem they literally meant 'Quite Enough Done'! A study of the geometry sections of many primary school mathematics books only confirms how much these experiences have influenced the attitude of the various authors.

If geometry is thought of as a study of spatial relationships it will be seen that whether or not primary school teachers teach it their pupils will certainly experience it. The child in the cot, as soon as it is able to start exploring the world around, is very much concerned with the relationships of size, shape and weight, and this concern continues all through life. An awareness of these relationships is the first evidence of mathematical developments, in fact, interest in quantitative relationships as expressed in numerical form does not develop until the child is about four or five years old.

In this connection the child is following the pattern of the mathematical development of civilisation. Man had a vital concern for size, shape and weight in building his house, making his clothes, hunting for food, etc., long before he found the need for measurement, numerical representation and computation.

Many infant schools exploit this natural spatial interest and make use of it for its numerical value, but tend to neglect its value from a geometrical point of view. The neglect is even greater in many junior schools. To a degree this can be accounted for by the traditional emphasis on arithmetic as opposed to mathematics in the attitude towards education in the past and the various forms of secondary school selection to-day. With the general liberalisation of the selection procedure, it is to be hoped that more schools will be encouraged to take a wider view of young children's mathematical needs and interests.

Much can be done in both the infant and primary school to make the young child more consciously aware of the many different sizes and shapes with which he comes into daily contact. In the top infant and lower junior classes a start can be made to arrange such objects systematically and consider them logically. The children can be encouraged to make a collection of empty household containers which each represent a group of articles of the same shape, e.g., those with six sides (rectangular prisms), those consisting of two circles joined at their circumferences (cylinders), those with a curved surface of constant curvature (spheres), etc. This leads naturally to the next stage—can a collection be made of containers each one having all its sides of the same size and shape (the regular solids)?

Making these collections will give rise to a great deal of discussion—how many sides are there?; are the edges all equal?; what sort of corners are there where the edges meet?; etc. This will do a great deal to improve the children's mathematical vocabulary, without which there can be little real understanding. Many of the words will already be partially familiar (bath *cube* and percussion band *triangle*), some will be familiar although not really understood (width, breadth, thickness), and some will be entirely new (perimeter and circumference). Junior children take a real delight in learning new words and have no fear for such strange-sounding words as tetrahedron and dodecagon. They should therefore be given the true geometrical names right from the beginning—rectangle rather than oblong—not only because they can 'take' them but for ease of understanding later in their school lives.

This discussion will quickly draw attention to the corners (angles) made by boundary lines (edges) where they meet. As this is so frequently a square corner it may be desirable to devote some considerable time to the right angle at an early stage—square corners, paper folded along one edge, the plumb line and spirit level and the compass (magnetic). The more practical work that can be done on the right angle the better—it will be a valuable investment for later work on angles.

Comparison of these three-dimensional objects will usually be confined to two dimensions at a time in the early stages; that is, one side of one object will be compared with one side of another. This gives a concrete introduction to the two-dimensional work with which so much of his later formal geometry will be concerned. There is a tendency to assume that a child appreciates three straight lines drawn on a piece of paper so that each cuts the other two, as being a triangle. But this may be far from true; young children find considerable difficulty in appreciating drawn outline figures of this sort. Surely the top of road warning signs, the ends of a ridge tent, a square or rectangular sandwich cut diagonally in half or a cycle pennant all have much more meaning and reality for the child from which the true conception of a triangle and its characteristics would be grasped with greater understanding.

Just as the children have made a collection of three-dimensional objects, so a similar collection of two dimensional figures can be made. Incidentally, all these collections should be mounted, suitably labelled and notes added to each item in keeping with the understanding of the children concerned. It will be interesting to see how difficult it is to find a square article that is in everyday use! The pentagon may also prove difficult—one may feel justified in showing the children how to make one by tying a single knot in a strip of paper and outlining the shape made by the overlapping thicknesses of paper. For some figures it may be necessary to take the top of a three-dimensional object (the top of a vanilla slice makes an excellent parallelogram) or ignore the thickness (a bronze 3d. piece for a dodecagon).

By cutting and folding these two-dimensional figures (or one side of a three-dimensional figure such as the top of a match box) the children can experiment under the teacher's guidance and thereby learn a great deal about their make-up. A diagonal fold quickly turns a rectangle into two triangles; can any triangle, therefore, be made into half a rectangle? A parallelogram looks like a squashed rectangle; what is the connection between a parallelogram and rectangle of the same length and breadth? A rectangle can easily be turned into two triangles; can all straight sided figures be broken down into triangles? Can a repetitive pattern be

made with any straight sided figure and what would it look like? Quite apart from its geometrical significance, all this foundation work will be of inestimable value when the time comes to introduce area.

From the overall shape of the figures the attention can next be turned to the corners. The children can be shown that by tearing off the corners of a triangle and placing them edge to edge around a point, the two outer edges will make a straight line. Experiment will show that this happens whatever the size and shape of the triangle. Treating the corners of various figures in this way can do much more to give a sound introduction to angles and various geometrical principles than any number of pairs of intersecting lines drawn on paper or other diagrams. With a bare minimum of help quite young children can discover that the sum total of the angles of any given figure is equal to the angles of all the triangles that go to make up this figure. Portions of cheese in a cheese box, slices of a circular cake, orange segments, etc., all lend themselves to further experimenting and discussion along these lines.

The work may now be extended: opening and closing doors, the side arms of a deck chair, the folding legs of a collapsible table, a pair of scissors, the hands of a clock, the indicator arms of a pair of scales and a car's speedometer, all these and many other similar everyday articles give ample scope for discussion on the fundamental idea of various sizes of angles and their properties. The work on the right angle already mentioned will need careful integration with this study of angles.

All too frequently children think of tin lids, buttons, coins, plates, etc., as circles. Instead, what could be more natural than to lead on to the circle from the concrete and practical consideration of angles. The figures described by the outer edge of doors and book covers, the tips of clock hands, the flight of tethered model aeroplanes give a much more useful geometrical representation of the circle, and provide invaluable discussion material. Radius, circumference, arc, segment, quadrant—all have real meaning in these forms and above all the centre of the circle is given its place of importance.

It would be appropriate here to mention the valuable contribution that meccano strip and similar constructional material can make. In making outline shapes of two dimensional figures with this equipment many of their properties and characteristics will be encountered. Figures of various shapes can easily be compared and broken up into their composite parts. Elastic bands stretched round nails driven into pieces of board can be used in much the same way, although for young children this is not as suitable as the rigid meccano strip.

The three-dimensional aspect of shapes can be treated in much the same way as outlined above for two-dimensional figures, although a greater degree of manual skill may be required for handling the material. For example, the gelatine base of used X-ray film is very useful for completing the shape of truncated pyramids (chocolate Easter egg containers) and cones (ice cream cartons). This material can also be used to make false sides when cutting up shapes to show their make up, or making shapes from 'nets' (sellotape is a convenient means of butt-joining the edges).

As scale, proportion and ratio play an important part in geometry as well as in arithmetic, it is worth spending some little time on this. Once again the use of

real objects can act as a stimulating basis for discussion and, with the older children, measurement. Miniature packets of cigarettes, household commodities in cartons, plastic clothes pegs, sample boxes of cereals and match boxes of different sizes provide excellent material for comparison and discussion with their normal sized version.

In all this work it would be unwise to say what should be done at any particular age. Some children will be ready earlier than others and progress faster because of their richer experiences, but depth of understanding should never be sacrificed for speed of progress. This type of teaching is very difficult to test. Much of the work can be repeated each year in the junior school making a deeper and more detailed study as the child grows older. For example, the idea of the parallelism of opposite sides of a square, rectangle and parallelogram may not be introduced until the second or third year in the junior school; this new characteristic of these shapes will then naturally lead to further discussion of parallelism as met in railway lines, ruled note paper, overhead electricity cables, etc.

There are many other aspects of geometry that could be used in the primary school without touching on the theorem of Pythagoras and similar matters which can be well left to the secondary school. Mathematical 'embroidery', the making of geometrical models from 'nets' and paper folding in all its fascinating aspects are several, but these are the preserve of the gifted enthusiast. This article has tried to confine itself to an approach within the ability of all teachers using material readily available to all children.

In conclusion, the aim of the primary school should be to provide young children with a sound foundation of knowledge and understanding of the spatial relationships with which he comes into daily contact and help to make him more alive to this aspect of the world around him. Without this foundation his future geometry is unlikely to have any real meaning, and may even become the 'learning of tricks' and their application, as is so frequently the case in arithmetic. It is on this foundation that the secondary school will later be able to build, using a more formal and abstract approach.

HOPE ABANDONED!

We have received a complete set of maths. books covering both primary and secondary schools. Unfortunately they are written in Japanese (at least there was a Japanese stamp on the parcel). If anyone would like to review them and knows the language, we should be very glad to hear about it.

C. HOPE.

(It may be of interest to Mr. Hope and to our readers to know that copies of our magazine are sent regularly to Japan, as well as to most other countries in the world.—EDITOR.)

THE GEO-SPACE

ANGELO PESCARINI, translated by J. G. Dixon

(This article appeared originally in "Orientamenti sulla didattica della Matematica nella scuola seconderia" published by Archiro didattico del Centro Didattico per l'Istruzione Tecnica e Professionale, Rome, 1958).

Some years ago, in the Scuola Media "Pietro Damiano" in Ravenna, I devised an apparatus for teaching three-dimensional geometry. The Headmaster, Ugo Serra, gave me enthusiastic help. By necessity, the apparatus was more complex than the geo-board; I called it, by analogy, the geo-space.

It is a hollow, transparent, perspex cube with edges 50 cm. long; one side can be opened, and it stands on a movable table. Swivel hooks are placed so as to form a square lattice on the internal faces (25 per face) to provide points of reference. Elastic threads with loops at the ends, so that they can be slipped on and off easily, are stretched between hooks to represent straight lines in space. These threads, and other accessories which are not described here, make possible a very clear, semi-abstract, representation of the sort of three-dimensional geometrical figures which are usual in the middle school. The apparatus makes it possible, at this level, to ensure the acquisition of a clear conception of spatial relations, from very simple ones to some that are really complex. The teacher exercises a psychological control: he can track down, and possibly remove, the difficulties of perception that the pupil has when faced with geometric three-dimensional situations.

It is well known that a geometric situation is not always perceived as a "coherent form" in the sense in which the term is used by psychologists (and particularly by the *Gestalt* school); it may be, however, that the geometric nature of the problem calls for uniform and simultaneous observation of the said form. Hence the perceptive tendencies of the pupil inevitably hamper the purely geometric consideration of the given form. We believe that the use of this apparatus helps the pupil to become aware of these perceptive tendencies so that he is freer and less restricted in his approach to the purely geometric considerations of form; thus he becomes better able to discern the possibilities of a given geometric situation.

In general, it may be said that the capacity for *re-structuration*, in the sense in which the word is used by Koffka (that is, the capacity to transform the structure of a mentally perceived figure into another) may be developed by appropriate exercise. We should also bear in mind that there may be a connection between the development of this faculty and of a purely geometric ability to look at a figure in such a way as to solve a certain problem as directly and simply as possible. A geometric proof, whether simple or complex, demands that this faculty of re-structuration should be well developed. In traditional teaching theory the attempt is made to develop this faculty by proceeding from the simpler to the more complex proofs. We maintain, however, that there is not always identity between the complexity of the mathematical logic required, and the structural complexity involved by the question, in a psychological sense. In other words, it is useful to separate the two factors, and to educate the faculty for structural analysis independently of that of mathematical proof. How many boys say that they do not see the way to a proof,

questioning their attitude to deductive reason, when in fact they have not "seen" because they did not know how to look and how to re-structure the form in the most suitable way?

For the same reason, even the judgment of the teacher may become arbitrary, so that he tends to avoid giving exercises which cannot easily be "seen". Hitherto we have had little ability to help forward this type of psychological development, which has been left to spontaneous growth.

These considerations are valid in general, but apply with particular force to three-dimensional geometry. Many secondary schoolboys do not succeed in visualising even the most elementary three-dimensional figures, and consequently cannot represent them in two dimensions; they ask for the teacher's help, thinking that it is a question of ability to draw. If we look closely, we find that the defect occurs from lack of intuition of spatial relations. "A geometrical figure", says Piaget, "May thus be the product of a logical construction, a pre-operational intuition, a perception, an automatic habit and even a building instinct. The difference between the various levels does not, therefore, depend on the content, i.e. on a 'pattern' somehow materialised, which results from the act, but on the 'pattern' of the act itself and its progressive organisation". This passage calls for comment, but I will say only that the geo-space was conceived within the framework of the ideas here described, and was inspired by my having used geo-boards.

The object is to develop operational intelligence*, which is of fundamental importance in the investigation and representation of space relations. Piaget intended, in the above remarks, to deepen "Gestalt" Theory: if they are true in that connection they are even more true in teaching theory, which contains so little to foster the indispensable education of spatial intuition.

A more ambitious use of the apparatus is possible to a teacher who knows how to use it for heuristic ends. A vast range of geometric situation can be exploited, by means of threads stretched in various ways to represent planes, solid figures, etc.

A cylinder can be constructed, with ends of perspex and generators of elastic, and using torsion one may produce from it the hyperboloid and the double cone so that it becomes clear that these surfaces can be considered as "ruled", or formed of straight lines.

Using a film projector and an exposed film on which a straight line is incised, it is possible to produce a lamina of light that is free to rotate with the chassis of the projector. The beam of light, directed to cut the generators of cone or cylinder, demonstrates the conic sections, which visibly transform themselves one into another in a thought provoking manner.

* For the meaning of this phrase see Piaget "Psychology of Intelligence" Routledge, 1951.

CORRESPONDENCE

Sir,

Mr. D. H. Wheeler's interesting article on pp. 24-27, 'Mathematics Teaching', No. 11, November 1959, dealing with the study of numerical sequences is useful and timely. A great deal of elementary algebra can be taught by means of this topic—and not only to Grammar School children. The fascination of attempting to find a formula—or a curve—to fit a given set of values can arouse a lasting interest; and, what is equally important, we have here a branch of mathematics where the pupil can make discoveries for himself and build up self-confidence in doing so.

May I make one point in connection with a statement in Mr. Wheeler's article? He writes: 'With some difficulty, a formula can be derived for the new series, . . . ' (1, 4, 10, 20, 35, etc). In point of fact, a formula for this series, and for all further series comprised in the Figurate Grid, can be arrived at with the minimum of difficulty. The formulae for the first three rows of values in the Figurate Grid are $1, n, \frac{n(n+1)}{2}$, which suggest, rightly, that each successive formula consists of a polynomial of one degree higher than that of its predecessor. If, therefore, we divide the given sequence by n, we should arrive at a formula which is of the same order as the formula for the previous sequence 1, 3, 6, 10, 15, etc., and which can be recognised.

We thus obtain $\frac{1}{1}, \quad \frac{4}{2}, \quad \frac{10}{3}, \quad \frac{20}{4}, \quad \frac{35}{5}, \quad \frac{56}{6}, \quad \frac{84}{7}, \quad \ldots$

which can be reduced to: $1, \quad 2, \quad \frac{10}{3}, \quad 5, \quad 7, \quad \frac{28}{3}, \quad 12, \quad \ldots$

giving all terms a denominator of 3 we get:—

$$\frac{3}{3}, \quad \frac{6}{3}, \quad \frac{10}{3}, \quad \frac{15}{3}, \quad \frac{21}{3}, \quad \frac{28}{3}, \quad \frac{36}{3}, \quad \ldots$$

The numerators of the fractions are obviously the sequence 1, 3, 6, 10, etc. 'stepped up one', for which the formula would be $\frac{1}{2}(n+1)(n+2)$. Incorporating the denominator 3, we obtain $1/6(n+1)(n+2)$. As we have divided the sequence 1, 4, 10, . . . by n, we multiply our formula by n and get: $\frac{n(n+1)(n+2)}{6}$.

We can deal with the sum of this sequence in the same way, eventually arriving at the general expression $\frac{n(n+1) \ldots (n+r-1)}{r}$.

Finally, the sequences comprising the Figurate Grid can be employed to find the formula for any polynomial sequence.

Yours &c.,

R. V. PARKER.

Mr. Wheeler replies:

'It is clear from the context of my remark about the difficulty of deriving a formula for the series 1, 4, 10, 20, etc. that I envisaged it as a rule for the number

of triangles formed by n lines in a plane. I suggested this approach because of the connection with the situation from which the 1, 3, 6, 10, etc. sequence had been abstracted. Mr. Parker is quite right in showing that this is not the easiest method or the one most suitable for generalisation. I was interested in his device of dividing the terms by the value of n in order to reduce the degree of the expression. I have usually by-passed this by using the direct comparison of terms in the two sequences. This, of course, comes to the same thing in the end.

Here, the ratios of corresponding terms are

$$\frac{1}{1},\quad \frac{4}{3},\quad \frac{10}{6},\quad \frac{20}{10},\quad \frac{35}{15},\quad \ldots$$

These are seen to be the same as

$$\frac{3}{3},\quad \frac{4}{3},\quad \frac{5}{3},\quad \frac{6}{3},\quad \frac{7}{3},\quad \ldots \quad \text{or generally } \frac{n+2}{3}.$$

So, multiplying a general term of the first series by $\frac{n+2}{3}$ gives a general term of the second.

Sequence	Ratio of term to one above	Ratio in terms of n	General term of sequence
1 1 1 1 1 ...			1
1 2 3 4 5 ...	$\frac{1}{1}\ \frac{2}{1}\ \frac{3}{1}\ \frac{4}{1}\ \frac{5}{1}$	n	n
1 3 6 10 15 ...	$\frac{2}{2}\ \frac{3}{2}\ \frac{4}{2}\ \frac{5}{2}\ \frac{6}{2}$	$\frac{n+1}{2}$	$\frac{n(n+1)}{2}$
1 4 10 20 35 ...	$\frac{3}{3}\ \frac{4}{3}\ \frac{5}{3}\ \frac{6}{3}\ \frac{7}{3}$	$\frac{n+2}{3}$	$\frac{n(n+1)(n+2)}{2.3}$

etc., etc.

We are obviously both agreed on the value of studying sequences in the classroom. I would like to stress again that the study can be of very great value at many levels, and not just the rather advanced one normally considered appropriate'.

The Editor would welcome letters from readers (not necessarily for publication) giving their views on the types of articles they would like to see in future issues and also on matters in general arising from the teaching of mathematics. He would particularly welcome correspondence on the points raised in the current issue by the various articles on Sixth Form Teaching.

STARTING A MATHEMATICS SOCIETY

JOHN PRICE

Although keen to do so I must admit that I was very hesitant about starting a mathematical society at the school. My main fear, I think was starting something that would eventually prove to be a flop after a very short time. I wondered whether it would be possible to run such a society in a Secondary Modern school and whether I, a newcomer to mathematics teaching, would have the ability to keep it going. Anyway, although the arguments against seemed overwhelming, I let indiscretion prevail and asked the head for permission to try out my idea. He readily consented and seemed quite pleased.

The next thing to do was to get some of the keener pupils in my first year 'A' class to suggest to me that such a club ought to be formed. This was more or less successfully accomplished and after saying to them what a good idea I thought they had, I put one or two if's and but's to them. "Would there be sufficient support?" They soon answered this by going round the class (I may have planted the idea) and returning with a workable list of names. "But if you want me to do this for you I would expect regular attendances!" "Oh yes sir." And so it went on. I said that I would have to think about it!

The following lesson I announced that the Mathematical Society would have its first meeting after school on the Thursday of the following week and would last for about 45 minutes. The subject would be "The Mobius Band". Fourteen turned up for the meeting which proved to be quite good fun and generally a successful meeting. This was immediately before Christmas. On returning the following term I was asked when the next "Maths Club" would be (they always call it by this name). This was the signal I wanted so I arranged fortnightly meetings throughout the term and charged them 3d a term to belong.

This arrangement has now worked satisfactorily for nearly two years. For their 3d they receive a copy of Mathematical Pie and are entitled to all the privileges of membership. All this means is that they can attend the fortnightly lectures, make use of the society's library (which they do incidentally, although we haven't many books in it yet) and go on the annual visit to a place of interest. So far we have only had one visit. For this the society had a whole day free from school to visit the aeronautical section of the science museum in the morning and the London Planetarium in the afternoon, the school paying all expenses. A colleague accompanied me and we had the full membership of 17 with us. I am hoping to arrange a visit to the Mint for our next annual outing.

The winter months when the evenings are foggy can be a difficult period as the youngsters ought to be home as soon as possible. However if it is excessively foggy I cancel the meeting until the next week. There is usually a groan or two at this, but it is better that way because half the members (usually the girls, quite naturally) are asking if they can go home, and total numbers aren't great in any case.

Although the sexes are evenly divided in the school the boys out-number the girls in the society, but we always have some girls.

The major problem is the subject matter for the ordinary fortnightly meetings. Bearing in mind that these are Secondary Modern children who may—I say *may*—

reach the dizzy heights of 'O' level, I find it rather difficult to continually find interesting topics of a sufficiently elementary standard. The greatest help that anyone could give me would be to supply a list of suitable, tried, topics. For the benefit of others I append a list of successful lecture subjects at the end of this article. Fortunately I have never been right out of a topic. Mostly I have two or three in hand (4 to 6 weeks). But once or twice I have had to do some hard thinking and looking in books for inspiration.

From time to time I persuade new members to join. Existing members sometimes introduce a friend but my greatest pleasure is gained when someone comes up to me and says "May I join the Maths Club sir?" I keep a register of attendances and encourage members to take a pride in the number that they make. Three or four stalwarts have attended practically every meeting. Although I run the society on my own, colleagues (including the headmaster) have willingly responded when asked to address the society. This gives me a chance to rest for a while and for the children to have a change of face and form of presentation. For instance a geography teacher gave three talks on astronomy and an Indian lady we had on the staff gave a talk on Indian coinage.

For many of the earliest meetings my artist wife prepared posters each time. I believe these have a limited value in bringing members to the meeting but they do let the rest of the school know what is going on. Now we have a publicity secretary whose job it is to prepare a poster and warn all members the day before the meeting.

Numbers at meetings have not been high. The average for the first year was 9.4 and for the second year is 9.9; but the main thing is that it is a workable number. The lowest I have had is 5 and the highest 14.

A difficulty I foresee for the future is the wide age range and consequent mathematical ability of members. I should like the members to form a committee with officers but I do not think they are ready for this yet. And perhaps the club should have its own buttonhole badge. Another idea I have toyed with is to change the society from a mathematical to a mathematical and scientific society and thereby appeal to a wider audience and get some assistance from the science side of the staff. Still one is continually feeling the way, and cannot always be certain of the manner in which things will develop. But whatever the future holds it is true to say that the past has been reasonably successful. These children are not frightened of mathematics and even those who are not members, I feel, benefit by seeing their classmates doing "spare time maths."

Lecture topics:

Napier's bones
Pi
Foretelling the age of the moon
Algebra
Polyhedra
Mathematics of Easter
Levers
Mathematical Puzzles
Mathematical curves
Comets and Meteors
How the calendar began
How today's calendar began in Rome
What the calendar may become
The vernier
The micrometer
Alice in numberland (film strip)
Estimation
Squaring the circle
Indian Coins
Ellipses and satellites
Secrets of the universe
Measuring the Earth.
Mobius band

THE CONCEPT OF AREA

G. P. BEAUMONT

Failure in Geometry is a major difficulty in the teaching of mathematics. Commonly, thirty to forty per cent of the candidates entered fail at Ordinary Level, many because of weaknesses in Geometry. Now this is the case in spite of the revision of the syllabus, the introduction of a 'Stage A', and attempts to jazz up the subject by exploring applications to surveying, scale-drawings and map-making. Nor has there been wanting anxiety, good-will and effort in the battle to improve the situation. How is it then, that we still fail to sell Geometry to our pupils? It seems to us that one of the causes is the comparative neglect of the psychological development of concepts in the minds of our pupils. Unless we attempt to gauge and match this development, much of our 'craftmanship in mathematics' will go for nothing.

This is not to imply that none of the findings of psychology are applied to teaching problems. They are indeed: but what aspects are seized upon? Usually those that tell us about efficient memorising, learning by association and the advantages of visual aids. We use psychology to help us make the work more palatable, whereas other parts that concern us are rarely discussed. Yet, in a paradoxical way, by dint of bitter experience, most class teachers find out a greal deal about how children think when they are doing mathematics, though such valuable information but rarely gets into print and may not even be passed on to colleagues. (There is even a danger that when we have received enough shocks in the classroom, we shall decide that, after all, too many of the children are not intelligent enough to profit from the work).

Now what are these situations that occur every day and should be the material for rethinking teaching problems? Well, some of the most important surely are:

(1) Inability to understand the meaning of new words.

(2) Inability to construct a diagram from written instructions and the eagerness to draw special cases.

(3) Conflicting images in the minds of the pupils and the teacher initiated by the same verbal forms.

(4) Parasitical dependence upon a particular orientation or lettering of a figure.

(5) Uncertain appreciation of concepts like *line*, *circle*, *area* and *volume*.

We commence our discussion of area with two quotations. The first is from a well-known text by Godfrey and Siddons: "A subject on which it is fatally easy for a class to work examples without much understanding". Yes, indeed; and the ease remarked upon leans almost entirely on the simplicity of the formula $A = lb$. Our second quotation is from a report by the Incorporated Association of Assistant Masters: "Area is a kind of surface size. It is more important to compare irregular areas than regular, since this emphasises the notion of measuring areas by covering them with a net. The notion that we find area by counting squares is fundamental". The word "squares" needs scouting. 'Find the area!', we demand, emphasising the square units to such an extent that it is small wonder that our pupils frequently

fancy that irregular figures have no area. How fortunate we are that we can offer our children the formula πr^2 to cover the non-rectilinear cases.

A word like "area" contains for us *an accumulation of our experience*. To explain it in a few words is rather like a Zen Buddhist exercise in which, say, a bucket has to be described without using its name! Yet we readily assume that our pupils' experience is congruent (nay, contained in!) our own. Such tamed concepts are capable of biting. We remember the shock received on first being shown the properties of the Mobius strip. Much of mathematics is recognition and pretending, and as Wittgenstein has remarked; "Hold a drawing of a face upside-down, and you can't recognise the expression of the face".

Now area is some sort of invariant of a figure. How shall we even know whether a figure has an area or not? What about gaps in the perimeter, or points missing from the interior? This is what Piaget has to say about invariance: "Every notion, whether it be scientific or a matter of common sense, presupposes a principle of conservation, either implicit or explicit, e.g. inertia, matter, object, . . . A collection or set is only conceivable if it remains unchanged, irrespective of the changes occurring in the relationship". Our readers will be familiar with that part of Piaget's work which seeks to discover the order of establishment of certain key invariants and the age at which they are adopted. A typical example is the pouring of a constant volume of liquid into containers of different cross-sections. At certain ages, a child may miss the conservation of volume altogether and only concentrate on changes in the height and the cross-sectional area.

What follows is based upon a set of lessons designed to discover how pupils suppose that the area of a closed irregular figure should be calculated. The pupils were in the second year of a Central School.

Q What do we mean by area?
A Length times breadth.
Q (pointing to figure) How do we find the area of this?
A It's the space inside.
Q I thought it was length times breadth.
A (various) It has no length — there isn't a length times breadth — *Oh! it hasn't got an area!* — it's only for rectangles and cubes — for a circle it's a circumference, it would be π times the diameter.
Q Is it a circle?
A No.
Q Has it got an area?
A Yes.
Q What do we mean by "it has got an area?"
A It's the amount of space inside.
Q How shall it be measured?
A Same way as a circle: draw inch squares in it.
Q I want to see how it is done. Who will volunteer?
(A boy marks off inches around the perimeter)
Now what?
A I'm going to count them.
(Eventually the class rejects just counting inches around the perimeter

and suggest constructing square inches, each with two vertices on the perimeter. They don't seem very worried by the gaps between a side and the perimeter, but finally discard the idea when they see how the squares overlap. At last, they strike upon—remember?—the grid of parallel lines. Such a grid is constructed).

Q What are you going to do now?
A Count the squares. (Not all the class agrees).
Q What about the bits left over?
A They are half-inch squares, so put the bits together.
Q When I count the squares does it come to more or less than the area of the square?
A Less
Q What's left over?
A Bits of squares.
Q But are they bits of squares?
(1/4 of the class thinks yes, 1/3 thinks no)
Can the bits together make squares?
A No. They are not squares; they are not straight,
(Subdivision of the grid follows)
Q Do we need to count the old squares?
A No. The middle ones are all right.
Q Is there anything left over now?
A Yes, We must halve it again.

In a subsequent lesson, after a good deal of haggling, it was agreed that the precise orientation of the grid did not matter.

Only the main thread of the discussion has been presented; the omissions were the usual requests to weigh or distort the figure. One seductive idea implied enclosing the figure in a circle and finding the average distance between the two perimeters!

The above discussion emphasises that much time should be spent on the basic ideas, eager though we may be to proceed to the problem-solving stage.

Now let us consider diagrams on the blackboard. Without them we should be lost, yet because of their static nature they may also hinder a genuine appreciation of a proof. Frequently we present a particular theorem with a figure that has constant orientation and lettering. In doing this, we are encouraging the dangers inherent in learning merely by association. Weaker pupils attempt to learn the figure and the letters and complain should the figure be rotated. The Gestalt psychologist Wertheimer, investigating the understanding of the rule for finding the area of a parallelogram, played havoc with a class by rotating the diagram. Wertheimer's book, "Productive Thinking", is full of interesting information as to how good learning takes place and how difficulties arise. He stresses the part played by the "gestalt" of a particular configuration; for example, the sides of the parallelogram may force a consideration of the enclosed area as composed of elementary parallelograms. Hence, in very deed, the area *is* length times breadth. The reminder is that we too readily assume that neither the addition of lines nor the permutation of parts affects the area of a figure. Does this seem too elementary? Yet, in a neigh-

bouring difficulty, which of us has not discovered a fifth former blithely proving the congruence of two triangles, one of which is completely contained in another?

In conclusion; the abundance of seeming errors made by our pupils may help us to discover how they think. Success in a mathematical situation may be not just a question of a correct form of words and the understanding of a particular logical framework.

TWO REASONS FOR TEACHING ARITHMETIC

IRENE G. YOUNGMAN

Notes from a Scheme of Work for an Infants' School based on Cuisenaire material.

Reasons for teaching arithmetic:

1. To solve more easily the problems of everyday life.
2. To develop thinking and reasoning powers to the full, to stimulate interest and to give alertness to the brain.

1. Everyday Problems

Obviously everyday life is of vital and absorbing interest to a growing child—therefore it is of primary importance to extend that interest by enriching the experiences already encountered and by offering as many new ones as possible.

The first years of a child's school life should be full of practical and active experience. Teaching theoretically to a mind empty of experience is a waste of time and a bore to the child. *Understanding through practical experience must come before formal teaching.*

Experience through action with the aid of concrete materials is bound to bring a fuller understanding and interest to the young mind.

2. Arithmetic for its own sake

Following much practical work, which gives a reason for learning arithmetic, children can soon learn that arithmetic for its own sake can be a stimulating exercise. As a result of working out problems (which should be presented as problems and not just 'sums'), reasoning powers are developed to the full and the brain given stimulation and interest.

How then does Cuisenaire satisfy the requirements of this scheme of work?

(a) *Practical work and Problems*

A process in arithmetic has two parts:—

(i) *The abstract arithmetic*—pure numbers.

(ii) *The physics or physical side*—whether it be money, apples, elephants, weights, measures or the unattainable stars in the sky.

The physics of the problem is provided for by the practical and problematic side of the teaching; the abstract arithmetic with the aid of the Cuisenaire Rods.

Thus, in this type of arithmetic, 'the rods' are not an end in themselves but the instrument by which the end is achieved. It is the instrument that bridges the gap between abstract and concrete, obviously not the only instrument that can be used, but to my mind the one (of those known to me) that achieves its purpose with least waste of time and effort. Tedious and frustrating counting is no longer necessary since the numbers are represented by one whole instead of a number of units.

e.g. After playing with a shop and counting out many pennies, a child can easily be taught that his problem can be worked out very much more quickly (and accurately) by using rods instead of pennies. The last step is of course to manage without rods or pennies, but that will come later.

(b) *Arithmetic for its own sake*

Anyone who has had a little experience with the Cuisenaire method must realise the tremendous stimulus of the material, both to teacher and child. That is, if it has been used as it is meant to be used.

(i) To encourage thought and experiment and not merely to get correct answers to sums by a set procedure.

(ii) To encourage answers to questions rather than to teach a set of facts.

If taught wrongly, a child can be able to manipulate rods in a given way and get many sums right without really understanding what he has done.

The law must be '*Asking not Telling*'. It must be also '*Patience and Waiting*'.

There is no quick way to Cuisenaire unless it is a dishonest way. It is so easy to go too fast and to tell a child what to do, so easy to think *for* a child (which does nothing except cramp his thinking).

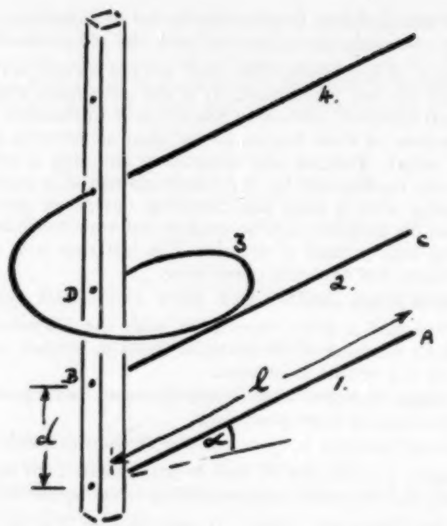
Pride must be set aside and we must learn to go at the child's rate of understanding—not at the rate that we think parents, critics, colleagues or pride consider necessary.

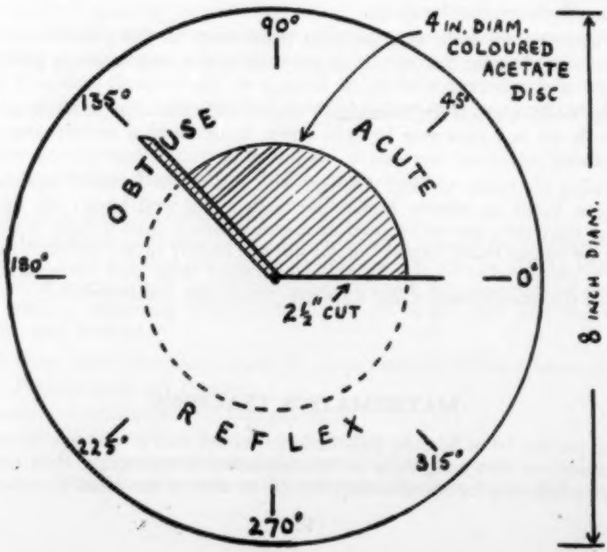
Because every child is an individual, with a different type of brain and personality, there is no law that can be laid down to demand a certain standard at a particular time.

In writing his books to help teachers with the method (much against his will) Dr. Gattegno faced an almost impossible task, as he well knew. He has made it amply clear that they are to be used as a guide rather than a rigid scheme. Some parts must be exaggerated, others cut, to suit the ability of the children in question, and there is plenty of scope for teachers to expand their own ideas as well as the author's—as the possibilities of the rods are practically unlimited.

MATHEMATICS TEACHING

is published on the 1st of March, July and November and is sent free to members of the Association for Teaching Aids in Mathematics. Membership costs ten shillings per year. Applications for membership should be sent to the Hon. Treasurer.


4.
3
C
2.
D
A
B
l
1.
d
α


90°
4 IN. DIAM.
COLOURED
ACETATE
DISC
135°
45°
OBTUSE
ACUTE
180°
0°
2½" CUT
8 INCH DIAM.
REFLEX
225°
315°
270°

TWO MATHEMATICAL MODELS

The Helix

D. J. LUMBARD

The model described here and shown opposite has been prompted by the inclusion in the Southern Universities Joint Board examination at 'O' level of a question on the lathe and also by a miscellany of text-book examples varying from 'wasps flying from jam-jars' to helical springs. Appreciation of the properties in 3 D is much simplified by the simple device shown here, which has definite advantages over the conventional 'cylinder from sheet of paper' technique. The cost of materials is small.

A piece of wood is drilled at intervals (d) at an angle x as shown. These holes must be of suitable size to allow plastic-coated curtain wire to pass completely through them. Lengths (l) of this wire, where $d = l \sin x$, are inserted in each of the holes so that the end is just half-way through the wood.

This simple two-dimensional arrangement gives all the required information, and the helix is formed merely by curving wire 1 and inserting A in B, curving wire 2 and inserting C in D and so on (as shown for wire 3).

The main virtue of this model is the ease with which it can be made and the very vivid impression of depth which the completed model imparts. I will gladly provide this model for teachers, who are shy of undertaking the actual construction.

Angle Model

I. B. REYNOLDS

The purpose of the model is to demonstrate that an angle is a measure of rotation. It consists of two discs of card of eight inches diameter stapled together with the top disc marked as shown and bearing a radial cut 2½ in. long to allow entry and exit of the coloured disc. This latter disc is made of coloured acetate sheet (20 thousandths of an inch thick) of diameter four inches. A ¼ inch hole is cut at the centre of the acetate disc, and a cut is made for the complete length of a radius; one side of this cut is then strengthened by fastening a narrow strip of acetate sheet along one edge to act as a 'pointer'. A brass paper clip fastens the movable acetate disc to the larger cardboard discs.

The acetate sheet is 'fed through' the 2½ inch cut in the top cardboard disc, and by rotating the pointer the model demonstrates the different types of angle. The centre (movable) disc could be made from coloured card, of course, instead of the acetate sheet, but probably would prove less durable in use.

(For the method of fastening the pointer to the acetate sheet disc, see the article *Simple Model Making in Plastic* in our last issue).

(*In our next issue there will be an article on simple model making for Primary School work. There will also be details of some mathematical models for advanced work*).

"AS I WAS SAYING . . . "

C. BIRTWISTLE

"Well, it's like everything else, it's over-rated. Now take this psychology lark for example; I can't say that I've got much use for it. And as for psychologists and psychiatrists, well . . . !

"Still there are times, you know, when it makes you think. Now take this sum for example:

$$1 \div 8 = \frac{8}{1}$$

Suppose you were marking it, would you say it was right or wrong?

"You would? Ah! but just a minute! I once remember reading a book by Cyril Burt—psychologist you know!—and he gave some examples of kids doing mirror writing. You know, they don't write properly; when they write, it's all the wrong way round; just like it is when you look through a mirror. They tell me that Leonardo da Vinci wrote like that, so you see it can happen to the 'A' kids as well as the 'C' stream. Well now, suppose it's one of those sort of kids that did this sum? So before you mark it you had better have another look at it, but through a mirror this time. Yes! Do it now!

"See what I mean?

"And then there's this chap on T.V.—Canasta or something he calls himself. Yes, you've seen him; does all sorts of tricks with cards, making people pick them out and saying what they've chosen, and then he says it's all done by psychology. Well, I don't believe it, do you? If you ask me, I think there's a bit of hanky-panky goes on!

"Anyway, they tell me you can do the same thing with numbers. I'll show you what I mean.

"I want you to think of a two-digit number between 1 and 50. Both digits must be *odd*, and they must not be alike, for example, you can't choose 11. Yes, go on; choose one now. Have you thought of one? Good!

"Now turn to that page in this magazine. You have? Right! Now read the first sentence at the top of the page. Yes, that's the one that starts 'Making these collections will give rise to a great deal of discussion'.

"All right! All right! Well, it seems that most people *do* choose that number; don't ask me why. If *you* didn't you must be one of those awkward types, so don't blame me!

"Anyway, you can test this idea much more simply than that. Just go up to somebody and say 'Give me a number between 1 and 10'. What? . . . Well, I *mean* somebody you know! Now it seems that most folk will say seven; again, don't ask me why. Of course, if you want to make it seem mysterious you can write the figure seven on a bit of paper before you ask. Yes, it's a bit daft if they don't say seven; all you can do is say 'thank you' and walk away!

"But I'll just give you another example. Yes, I'll try it out on you, just as I did with the number between 1 and 50. Ready? Now I want you to think of a two-

digit number between 50 and 100, but this time I want both digits *even*. Again the digits must not be the same; you've thought of one? Good! Well there are just so many pages in this issue of *Mathematics Teaching* (excluding the covers).

"Well, there you are, it all goes to show that you . . . What? It didn't work? There now, what did I tell you? All I can do is say 'thank you' and slink away! Can't say I think much of this psychology lark myself! Think it's over-rated! If you take my advice you'll stick to your maths!"

PIONEER MATHEMATICAL FILMS

TREVOR FLETCHER

News that a set of mathematical films had been made, some covering advanced topics such as the circles of curvature of an ellipse and cubics through nine points would delight all who believe that the film has a part to play in the teaching of mathematics. But this news is nearly fifty years out of date. The films were made in 1912. The early developments of the mathematical film which took place in Germany just before and after the First World War seem to be completely unknown in this country and it is quite possible that none of the films which were made there at that time have ever been shown here. A study of the synopses of these pioneer films teaches much about the choice of suitable subjects, reading appraisals of them shows the value of a constructive school of criticism, and contemplating the complete oblivion into which they have fallen proves once again the need for the effective exchange of learned films and the provision of adequate library and archiving facilities.

The principles of cartoon animation are incorporated in many nineteenth century optical toys, but the invention of the film cartoon is generally ascribed to Emile Cohl, of France, in 1908. Most of the earliest workers on cinematography were interested in the scientific aspects of the subject and they looked on the cine-camera as a means of analysing movement, of making scientific records rather than as a means of producing popular entertainment. Munch, of Darmstadt, was in this scholarly tradition, and within four years of Cohl's first cartoon work he demonstrated no less than eight mathematical films to a meeting of teachers in Halle. When we consider the time it takes to produce such films it is clear that the first ones must have been made almost as soon as the craft of film animation had been invented.

The first film on Munch's programme at Halle showed a proof of the Theorem of Pythagoras. This theorem does not seem all that suitable as a subject, but it has been filmed at least seven times by workers in various countries, most of whom were probably ignorant of the versions which had been made before. Films are laborious to make and such duplication of effort can only be deplored. It has been said that no experiment is entirely wasted since it either shows that something can or can't be done; but an experiment is wasted if it is merely the unnecessary repetition of a previous one. There has been unnecessary repetition in the production of mathematical films because of the inadequate circulation which such experiments have received.

Other films which Munch showed seem to have been of a type now associated with Nicolet, of Switzerland. There were films showing various positions of a pair of tangents to a circle, common tangents to two circles and the beautiful construction of the ellipse and hyperbola as the loci of the centre of a variable circle passing through a given point and touching a given circle. This last construction has recently been made by Nicolet and Motard, and the generation of the hyperbola is most striking as the two separate branches of the curve are produced by a continuous movement.

Munch also made a film on the problem of Apollonius for three circles, one on planetary motion, and the two on the advanced topics mentioned in the opening paragraph.

Shortly afterwards Detless was working on similar problems. He too seems to have made a film on the Theorem of Pythagoras; he also produced flicker-books of moving mathematical diagrams and wrote in the German educational press (1) on the use of such aids in teaching.

Developments were then interrupted by the War, but in the years immediately following at least two studios were making films. Deulig and Lichtverlag each made at least three before 1922. The Lichtverlag films were directed by Roseler and Schwerdt and receive particular praise from contemporary critics (2). One of their films showed pole and polar, another showed varying degrees of contact of a circle and an ellipse with the circle of curvature and the evolute (an excellent topic, in which movement can reveal a lot) and the last was entitled In-circle, E-circle and Altitude Theorem. This showed the centres of the inscribed and escribed circles constructed by varying the radius of circles until a fit was obtained, corollaries of the construction, and the original triangle as the pedal triangle of the triangle formed by the e-centres—from which the concurrence of the altitudes may be deduced.

It is only possible to judge these films from the synopses; but it is quite certain that the 'scenarios' show a fine appreciation of what a film diagram can do, and that they are still unsurpassed. We cannot speak of the technical quality—although it appears that they were all professionally made—but in any case an experimental academic film should be judged more for the vitality of its ideas than for its professional finish. Technical audiences are, if necessary, prepared to meet a film half-way and to try to see what it has to give. One cannot condone low standards in academic films—but experiments in teaching and in documentation are worth preserving irrespective of the technical quality, and they should be distributed internationally and copies should be readily available years afterwards as copies of learned books are.

The work in Germany at this period must have benefited greatly from the perceptive criticism which was appearing in contemporary journals. It seems that more articles were written about mathematical films in the early twenties in Germany than are being written in the whole of Europe at the present time. The key question of the balance between rigorous proof and appeal to intuition in a film received vigorous discussion, as did the question of whether or not films should be used to introduce a class to a topic or only be shown when the class already has some familiarity with it. We do not know the answers to these questions even yet; but it is quite certain that the teaching of mathematics with films would benefit if teachers considered these fundamental issues more thoroughly. Also, some writers emphasised

the role which the films can play in the training of observation; and this point is certainly not appreciated sufficiently today.

Max Ebner (3) shows particular insight into the scope and limitations of the medium, and his discussion of the special part which the film can play in mathematics is summarised in four main principles:—

i) Mathematical films should aim to help when the powers of thought or of visualisation meet difficulties, and should not seek to supplant the logical thought which is also necessary in mathematical instruction.

ii) They should be used when exact drawing with chalk is difficult technically, or too tedious, and should assist and develop the visual powers.

iii) They should only present material which cannot be presented better in other ways.

iv) The film should not seek to present artistic or surprise effects, but confine itself strictly to those which arise directly from the mathematics.

Allowing for a certain roughness in my translation it would be very hard to improve on this as a statement of the underlying principles.

When this work was done there were few projectors in schools, and few libraries of educational films, so it is understandable that such early experiments were not shown very widely and should not have received immediate support from teachers who had been trained in the old tradition. But though it is understandable it can only be regretted that films whose mathematical value may well have been great and whose historical interest it is impossible to deny have become unavailable to later generations. In the last twenty years several men have experimented with films like these in different parts of the world. More often than not they have proceeded in ignorance of what had been achieved elsewhere, repeating one another's experiments and repeating previous mistakes. Indeed, the mathematical film has been 'invented' by many men in many places.

This state of affairs is avoided in most branches of science by the free exchange of books and papers and by the maintenance of adequate library facilities. With films the situation is less fortunate.

Scientific and educational films fulfil the same function as technical papers, but international barriers which do not apply to books apply to films. These questions apart, there are also economic difficulties to overcome. The production of films costs money, and the film libraries are, from the point of view of the producer, too efficient. A film of minority appeal may be of great interest to a number of small, restricted audiences in different parts of the country, but whereas all Universities and their constituent colleges have book libraries of their own their needs for films are met from one central source. Few educational or research funds are prepared to sponsor films, but in many cases the production of specialised films could become self-supporting if there were more channels of distribution both on a national and an international scale.

The situation could be improved at an international level if the interchange of films was less beset by Customs regulations and red tape. At a national level universities should develop film libraries of their own, accepting an obligation to preserve scholarly films from all over the world, as they now preserve scholarly books. Furthermore these libraries should provide viewing facilities without fuss and formality

for small audiences and even for individuals. (16mm movie viewers are no more expensive than the microfilm readers which are normal library equipment).

Without support of this kind the learned film will never flourish, and forty years from now the results of today's experiments will, in their turn, be lost and forgotten. Unless suitable provisions are made now, in time to come ignorance of the creative work which is being done today will be as great as our ignorance of the pioneer work in Germany forty years ago.

References

1. Zeits. f. Math. u. Naturw. Unterricht, 43, 1912, p 440.
2. Detless. Die Veranschaulichung von veranderlichen Figuren in Unterricht. Unterrichtsblatter f. Math. u. Naturw., 1913, p 39, p 121.
3. Ebner, M. Mathematische Lehrfilme. Zeits. f. Math. u. Naturw. Unterricht, 53, 1922, p 193. (This contains references to many other articles.).

FOOTNOTE: Whilst Emile Cohl's place in the history of the animated film is firmly assured it does seem that he was anticipated by Stuart Blackton, an Englishman, who made *Humorous Phases of Funny Faces* in U.S.A. in 1906 and *The Magic Fountain Pen* in 1907. The first of these films inspired Munch and the second inspired Cohl.

However, the main reason for this note is something more important. As we go to press we learn that much of Munch's work still survives! It has been found in Germany and the National Film Archive in this country is taking steps to preserve this unique material and arrange some of it in a form in which it may be projected on present-day equipment.

BIGGER AND . . .

This is the first of our new enlarged issues of *Mathematics Teaching*. We hope you enjoy it and we can promise you more exciting issues of the magazine in the coming year. But remember! Subscriptions expired on December 31st last, and this is the last issue you will receive unless you have paid your subscription for the present year. Don't risk missing what we have in store for you; send your ten shillings to the Treasurer *at once* before you forget.

A FEW WORDS OF THANKS

Increased size means increased costs and this must be met by increased membership. May we thank all our members who have helped in recruitment during the past year, and hope for further efforts to increase our membership during the present year.

In the production of this latest issue we must express our thanks to our contributors and translators for their willingness and enthusiasm, to our printers, Messrs. Coulton & Co. for their ever-ready co-operation and advice, and to the staff and students of the Drawing Department of Nelson Secondary Technical School for assistance in preparing some of the diagrams to illustrate the articles.

COMPETITION

A new school is being built and the mathematics staff have been called in at the initial planning stage to advise on the siting, design and equipment of the mathematics room (this is an imaginary situation, of course!). Readers are asked to submit their ideas in detail to the Editor, stating the type of school (Junior, Secondary Modern, etc.) which they have in mind. A prize of a book-token for one guinea will be awarded to the best entry which is received, and this entry will be published in a future issue together with a review of ideas submitted in the competition. The Editor's decision is final. Last date for receipt of entries: April 30th, 1960.

NEW APPARATUS AND MATERIALS

RECENT FILMS

Three more mathematical films have recently become available. The first two are both versions of the Theorem of Pythagoras. A version of the film produced in Yugoslavia under the direction of Milan Krajnovic is now distributed with an English sound-track by Gateway Films Ltd., 470 Green Lanes, Palmers Green, London. It is in two parts each lasting about five minutes. The first part is essentially an animation of Euclid's proof using deformations and rotations to emphasise the equal areas involved; and the second part shows the proof of the cosine formula by similar methods.

Stuart Wynn Jones' four-minute film, again in black and white, covers the same ground as the first part of the other film, but treats it in a completely different manner. The draughtsmanship is in a contrasting style and the mood is entertaining and almost flippant. Abstract sound is used as a geometrical convention. This has not been developed in mathematical films anywhere before. The cinema, which was international until the arrival of the 'talkies', has still to explore the many possibilities of abstract sound; but there is a particular challenge to teachers in seeing it used as a teaching aid and not merely as chi-chi decoration in an avant-garde experiment. Enquiries can be sent to Mr. S. Wynn Jones, 107 Fellows Road, Hampstead, N.W.3.

Abstract sound occurs again in Norman Maclaren's film *Rhythmetic*. Made in Canada, a 16 mm copy of this colour film is now available from the British Film Institute, 164 Shaftesbury Avenue, W.C.2. Those familiar with Maclaren's work will know that it defies description, so we will merely say that in *Rhythmetic* numbers

appear and gradually spread until they cover the screen. They move about, they do sums, and they develop personalities. This most emphatically is *not* a teaching film, but more than any other film we have it shows the freshness of mind and the imagination which mathematicians need. I have shown this film to an audience of teachers, and I regret to say that half of them took it straight and were not amused. I showed it three days later to mathematical students at University and they roared with abandoned laughter for most of the time. What is it that these years in the class-room take away?

DR. STERN'S STRUCTURAL ARITHMETIC APPARATUS

Manufactured by Educational Supply Association. Complete set of six items £7 18s. 10d. plus Purchase tax £1 9s. 8d. (individual items may be bought separately).

Children Discover Arithmetic. Dr. C. Stern. Publ. Harrap 25s.

Few British teachers will have had the opportunity of using Dr. Catherine Stern's materials for developing children's ideas of number. Until recently the apparatus could only be obtained from the U.S.A. on an import licence. Thus E.S.A.'s new venture is particularly welcome. As neither the work books nor the teachers' handbooks are available, a copy of Dr. Stern's book is necessary if the materials are to be used in the way she intends.

In Chapter 1 the author points out that attempts at enlivening number work often result in the children being distracted by the nature of the concrete objects themselves; hence the significance of the number relationships is not understood. Dr. Stern claims that her materials have all the properties of real numbers and that their use will lead to an understanding of number operations and of the number system.

The apparatus is strongly constructed in wood and hardboard and consists of:

1. *The Ten Pattern Boards* with recessed squares to take ¾ inch cubes. The patterns are arranged in Montessori groupings.
2. *The Unit or Ten Case* and the blocks which fill it. These are based on the ¾ inch cube. There are two each of the blocks one to nine, and one ten block only. They are marked to show the units present, but brightly coloured to aid recognition without necessarily counting.
3. *The Counting Board* has ten grooves to fit the blocks from one to ten in that order. It has detachable number labels and a detachable number guide.
4. *The Number Cases*. These are square trays which stack inside one another. Each can be filled with blocks to give the various compositions of a number, e.g. the four case holds the three block and the one, two and two, one and three, and one four block alone.

5. *The Number Track*. This comprises ten sections, each ten units long, with a trough to hold blocks. The sections can be jointed together. The sides are marked from one to a hundred. A wire jumper can be moved up and down the track. (In the set received for review it was found that the sections were stiff and difficult to fit closely together, causing errors when large numbers were measured. It is to be hoped that the manufacturers can remedy this).
6. *A Box of a Hundred Unit Cubes*. These are for use in the pattern boards or in the number track.

The use of the equipment can be varied to suit the development of individual children. At the earliest stage children play number games involving estimating and measuring the blocks and recognizing the number patterns. Number names are next introduced, then written symbols. Later, children make records of what they have seen and done.

The children discover the facts about the apparatus in three ways:

(a) In groups of about ten with some guidance from the teacher.
(b) In free play.
(c) In groups of three to five playing simple games with a group leader.

The materials make clear the cardinal and ordinal aspects of number. The concepts of conservation and the reversibility of operations can be demonstrated in many ways by means of games. Odd and even numbers are shown clearly and many number bonds are learned incidentally. When Piaget's Stage Three of understanding has been reached and the composition of numbers up to ten is known, the 'teen' numbers can be studied in the number track. It is a pity that the 'twenty' cases are not available to give greater variety of experience at this stage. Multiplication, division and tables can be learnt by using the number track. The Dualboard is another piece of useful equipment which has not been made available in this country; it is used to explain positional notation, carrying and borrowing.

However, the essential materials for developing the basic number concepts are now obtainable. They are attractively coloured and children enjoy the activities of fitting, matching and measuring with them. The reviewer has found that normal infants gain a sound ground work, so that they are able to proceed confidently to written work without recourse to counting fingers or dots in order to arrive at a total. Backward seven-year-olds who have been asked to do formal sums before they have developed the necessary basic concepts have gained understanding of these concepts; they have come to enjoy arithmetic, and have seen the sense behind the conventions of notation. Their work is accurate.

Although more expensive and more complex than other structural materials, the Stern apparatus is largely self corrective. It can be used with a minimum of teacher guidance in large classes of very varied ability. It gives a sound basis for later mathematical work.

Its use in Infant and Junior schools should do much to make arithmetic a subject to be enjoyed and understood.

P. H. ELLIOTT

BOOK REVIEWS

Mathematiques et Mathematiciens

P. Dedron and J. Itard, Magnard, Paris, pp. 433.

This book is in two parts; the first studies the history of mathematics from the earliest times until the end of the eighteenth century, and the second goes more deeply into particular questions such as notation, equations of the first and second degree, and the classical problems which are not solvable by ruler and compasses.

The illustrations are excellent throughout. There are portraits of many mathematicians, reproductions of pages from famous books, and pictures of the buildings, paintings and scientific instruments which were the cultural setting within which mathematics developed. This very French feeling for *civilisation* and *culture* makes this a history unlike any which we have in England. The book is expressly concerned with 'elementary mathematics', but the treatment is never superficial. Original authors are quoted extensively, and the book demands continuous care and thought from the reader. For example, Egyptian and Babylonian numeration and calculation are discussed in some detail; there is a discussion of the difficult books of Euclid on irrational numbers; and there is a very instructive account of the way in which the notions underlying the calculus were made precise by the successors of Newton and Leibnitz.

This book will surely have great success in France as a school prize, and teachers elsewhere who acknowledge a responsibility to present mathematics as a chapter in the cultural history of man will find in it much inspiration. T.J.F.

Examples in Engineering Mathematics for Students (Book 2)

Vesselo and Glenister, George G. Harrap and Co. Ltd. pp. 127. 6s.

The popularity of Book 1 which covered first-year National Certificate Courses has encouraged the publishers to complete a series of books which will cater for a three-year course in Mathematics for the ordinary National Certificate in Engineering. As the syllabuses for N.C. Courses vary greatly in the various technical colleges throughout the country the difficulty of compiling a syllabus which will satisfy all lecturers has been realised, but an excellent effort has been made to satisfy the majority of schemes.

Further much effort has been made to arrange the exercises in a suitable sequence and here the lecturer should find the book most beneficial.

The contents cover from simultaneous equations to applications of Differentiation with a final chapter on Maxina and Minina.

Each section has worked examples, clearly explained, and this should prove an excellent help to the student. The grading of examples has been carried out with care, and to the reviewer, the entire work was stimulating and will provide an answer to many worried Technical College lecturers who may be searching for the text book which will meet the needs of the more harassed students. H.F.

Problems in Arithmetic

H. B. Beech, A. & C. Black Ltd., Books 1, 2 and 3, and Teacher's Books 1, 2 and 3. (Each 64 pp. except Teacher's Book 3, 80 pp.)

These books provide a detailed scheme of work designed to lead children to approach the solution of problems in an orderly logical manner. Starting with problems involving only one step of addition or subtraction of number the children are gradually faced with greater complexity of units in problems requiring 3 or 4 processes for their solution. Some 1,500 problems are carefully arranged to give a steady gradation. Unfortunately the author remains entirely within the conventional arithmetic of the Primary School course and does little to give the children any mathematical experience of significance. Further the Teacher's books seem to envisage imposing on the children a rigid framework within which to work at a time when they should be encouraged to experiment freely with their own ideas.

D.T.M.

Practical Metalwork.

An introductory course. P. C. White. G. G. Harrap. pp. 112. 9s. 6d.

The author has set out to establish a carefully graduated course in metal work and on the whole basic skills are built up within its framework with a few notable exceptions.

The child in his first year of metalwork is already faced with the task of cutting and shaping a new material and to achieve the degree of accuracy demanded by such models as calipers, straight edges and grinding gauges at this stage is inadvisable.

Whilst it may be accepted that the majority of these models have stood the test of time this book has nothing new to offer. The designs that the author has adopted and his choice of material will tend to date this book: why, for example, select tin plate for a fish slice, book racks from brass and drawer handles from mild steel?

The inclusion of tin plate work is unfortunately inevitable in books on metalwork and I feel that a "new look" on this aspect is desirable. Indeed, many craft teachers have dropped it entirely as the results have rarely been satisfying. It may be argued that by so doing valuable experience in development problems are lost, but this may be largely compensated by expansion in the field of decorative work and of course in the technical drawing syllabus.

The drawings, whilst comprehensive do not comply strictly with B.S.S. specifications. In particular I do not like the outline and dimensioning. Rather like the man who moves behind the bowler's arm, the dimension lines distract the eye from the main outline and too often are crowded inside the job. A bolder outline and more care in spacing is indicated.

Finally I feel the author has attempted to cram too many aspects of metalwork in one volume, in particular the inclusion of decorative metalwork with so little emphasis on design is undesirable.

J.H.

A.T.A.M. DIARY

1960

March 5th—Middlesex Branch. *Mathematics in Modern Schools.* at Aylestone School, Wimbledon.
Details—Miss J. Blandino, 9, Barnhill Rd., Wembley Park.

March 5th—Bristol Branch.
Details—J. Lumbard, Wenvisue, Chandag Road, Keynsham, Nr. Bristol.

March 8th—North-East Branch. *Spatial aspects of maths. in the Junior School,* at Education Dept., King's College, Newcastle.
Details—B. P. O'Byrne, 32, Teviotdale Gardens, Newcastle-upon-Tyne, 7.

March 12th—Colchester. *Mathematics in the Primary School.*
Details—Mr. P. Carpenter, Cambridge Institute of Education, 2, Brookside, Cambridge.

March 18th/20th—*Week-end Conference and Annual General Meeting* at Redman's Park House Hotel, Blackpool.
Details—C. Birtwistle, 1, Meredith Street, Nelson, Lancs.

March 26th—Cambridge Area—*Notation.* Manor Secondary School, Cambridge.
Details—N. Reed, Swavesey Village College, Swavesey, Cambridge.

April 2nd—E. Suffolk Branch—Saxmundham.
Details—T. Hampton Wright, Saxmundham Modern School.

May 14th—Crawley.
Details—Mrs. A. Ogle, The School House, Rogate, Petersfield, Hants.

May 21st—Middlesex Branch—*Top and Bottom: Primary School and Sixth Form.*
Details—Miss J. Blandino.

PUIG ADAM

As we go to press we have received news from Spain of the death of Puig Adam. We hope to pay tribute to this great teacher in our next issue.

MEMORIAL TO JACK PESKETT

A donation of £14 has been sent to the Imperial Cancer Research Fund as a memorial to Jack Peskett. We would like to thank those members who contributed.

A.T.A.M. BOOK ADVISORY SERVICE

Because of the cost of acquiring a reasonable number of suitable books, the Association has decided to discontinue its Library service. In its place, a Book Advisory Service is being instituted which will answer members' queries and recommend books for use and study. It is not intended to produce a comprehensive bibliography, as this has already been done by other bodies (e.g. a book list has been produced recently by the Mathematics Section of the A.T.C.D.E., obtainable, price

2s. 6d., from 53a, Brewer Street, London. W.1.), but to answer requests for information about books on specific topics.

Members who wish to use this personal service should write, enclosing a stamped addressed envelope and as precise a description as possible of the field in which they are interested, to Miss I. L. Campbell, City of Worcester Training College, Henwick Grove, Worcester.

FILMS

Equipment and advice is available for the making of mathematical films. Write, enclosing a stamped addressed envelope, to Mr. T. J. Fletcher, 148, Lennard Road, Beckenham, Kent.

MODELS

Models may be made to readers' specification for the cost of the materials involved. Write, enclosing a stamped addressed envelope and giving details of requirements, to Mr. D. J. Lumbard, 'Wenvisue', Chandag Road, Keynesham, Near Bristol.

CONFERENCES

The Association is willing to arrange Day Conferences for teachers of mathematics. It is usual for these to be sponsored by a Local Education Authority or an Institute of Education. Any member wishing to have such a meeting in his area should write to the Conference Secretary (address on back page) giving particulars of requirements and the amount of local support likely.

A.T.A.M. exists to bring about the improvement of mathematics teaching; this must be done through the teacher himself—and that means you! We shall be only too happy to assist you in your work. A few of the ways in which we can help are set out above. If, however, you have other enquiries, please write to the Hon. Secretary (address on back cover). In return might we ask you to help us? We depend entirely on the subscriptions of members of the Association for the production of this journal and all our other work. We can only extend our work and improve this magazine by increasing our membership. During this year we would like to increase A.T.A.M. membership by a half. Would *you* help by bringing the Association and this journal to the notice of other teachers of mathematics and encourage them to subscribe to it? New subscriptions should be sent to the Hon. Treasurer; membership costs ten shillings per year.

SUBSCRIPTIONS FOR 1960

Subscriptions for the current year became due on January 1st. If you have not already paid your subscription for 1960, will you please do so at once! Prompt payment assists the work of the Treasurer and the Association as a whole. Payment should be made to the Hon. Treasurer (address on back cover).

Our Next Issue . .

- ★ **Algebra in the Sec. Modern School**
- ★ **A Computer for Educational Purposes**
- ★ **Teaching and the New Mathematics**
- ★ **Measurement in the Junior School**
- ★ **Models for Primary Schools**
- ★ **The Function Concept**
- ★ **Blackpool Conference Report**

DON'T MISS IT

ON JULY 1st

ASSOCIATION FOR TEACHING AIDS IN MATHEMATICS

COMMITTEE

(*subject to alteration at A.G.M.*)

ELECTED MEMBERS 1959-60

President:	Dr. C. Gattegno
Vice-President:	R. H. Collins
Chairman:	I. Harris, 122 North Road, Dartford, Kent.
Deputy Chairman:	R. H. Collins, 97 Chequer Road, Doncaster, Yorks. J. V. Trivett, 16 Southwood Drive, Bristol, 9.
Honorary Secretary:	D. H. Wheeler, 318 Victoria Park Road, Leicester.
Honorary Treasurer:	Miss B. I. Briggs, 8 Glencairn Crescent, Edinburgh, 12.
Editor:	C. Birtwistle, 1 Meredith Street, Nelson, Lancs.
Director of Studies:	C. Hope, 68 Malvern Road, Powick, Worcs.
Conference Secretary:	D. T. Moore, 234 Birling Road, Snodland, Kent.
Publicity Secretary:	A. Ivell, 22 Gobions, Basildon, Essex.
	Miss Y. B. Giuseppi, 34 Pollard's Hill S., London, S.W.16. T. J. Fletcher, 148 Lennard Road, Beckenham, Kent.

REPRESENTATIVES & CO-OPTED MEMBERS

Primary Schools:	Mrs. A. Ogle, The School House, Rogate, Petersfield, Hants.
Middlesex:	Miss J. Blandino, 9 Barnhill Road, Wembley Park.
North-West:	Miss D. Clutten, 14 Bristol Street, Burnley.
West Riding:	B. Atkin, 212 Langwith Road, Langwith Junction, Mansfield, Notts.
Cambridge Area:	N. Reed, Village College, Swavesey, Cambs.
E. Suffolk:	A. S. Hampton-Wright, School House, Theberton, Leiston, Suffolk.
Bristol:	D. J. Lumbard, Wenvisue, Chandag Road, Keynesham, Nr. Bristol.
North-East:	B. O'Byrne, 32, Teviotdale Gardens, Newcastle-upon-Tyne, 7.
	G. P. Beaumont, 24 Birchington Court, West End Lane, London, N.W.6.